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A new optimum statistical estimation of the traffic intensity parameter for the $M/M/1/K$ queuing model based on fuzzy and non-fuzzy criteria

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Abstract:

Queueing theory has applications in various fields, including communication and computer design. Recently, there has been a growing interest in statistical inference related to stochastic processes, particularly in estimating queue parameters such as arrival rate, service rate, and traffic intensity. This paper focuses on estimating the traffic intensity parameter in the $M/M/1/K$ queuing model, where inter-arrival and service times follow exponential distributions with parameters λ and μ, respectively. We evaluate traffic intensity using Bayesian, E-Bayesian, and hierarchical Bayesian methods, applying the entropy loss function and suitable prior distributions for the independent parameters. Additionally, the shrinkage-based maximum likelihood estimation method is utilized for parameter estimation. A decision criterion based on a cost function and the Average Customer Satisfaction Index (ACSI) is introduced to select the most appropriate estimation, emphasizing those with higher ACSI values. To enhance understanding, we validate our estimations through the Monte Carlo simulation method and present two numerical examples based on the ACSI index.

Keywords: $M/M/1/K$ queuing model, traffic intensity parameter, cost function, ACSI.

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1. Introduction

In many practical situations, researchers often have prior information about an unknown parameter in the form of an initial guess value. Based on this guess value, a class of estimators known as shrinkage estimators is generated. These estimators were first introduced by Thompson (1968) in the following form:

$$T = m\hat{\theta} + (1-m)\theta_0, \quad 0 < m < 1. \tag{1.1}$$

Here, θ_0 represents the initial guess value of θ, and $\hat{\theta}$ is any ordinary estimator of θ. The coefficient m is also referred to as the shrinkage coefficient, which is determined by the researcher based on their belief about the initial value of θ_0. Various methods for determining the value of m have been presented in Kiapour (2018). In this article, m is determined in a way that minimizes the risk of the given estimator in Eq.(2.3). Values of m close to one indicate that the estimator tends towards the sample, while values close to zero indicate a tendency towards the initial value. If the initial value of the parameter is close to its true value, shrinkage estimators perform better than regular estimators, such as maximum likelihood estimation (MLE) (Kiapour (2018)). Shrinkage estimators have been applied in various scientific fields, including estimating average survival time in epidemiological studies (Harris and Shakarki (1979)), estimation in mapping studies (Wooff (1985)), predicting capital (Tso (1990)), and estimating mortality rates (Marshall (1991)).

Several authors have utilized shrinkage estimators in estimation theory. Prakash and Singh (2006) derived inverse dispersion estimators for the inverse Gaussian distribution using the Linex loss function. Singh *et al.* (2007) derived shrinkage estimators for the shape parameter of the Pareto distribution under the Linex loss function. Singh *et al.* (2008) derived Bayesian shrinkage estimators for the failure rate and reliability function in the one-parameter exponential distribution. Prakash (2009) derived Bayesian and Bayesian shrinkage estimators for the shape parameter of the Pareto distribution under the entropy loss function. Alhemyari and Al-Dabag (2012) derived a family of shrinkage estimators for the shape parameter of the Weibull distribution. Salman and Shareef (2014) conducted a preliminary test of the Bayesian shrinkage estimator for the scale parameter of the exponential distribution using the quadratic loss function. Kiapour (2018) derived classical shrinkage and Bayesian shrinkage estimators for the Rayleigh distribution under censored data.

The Bayesian method is one of several methods used to estimate the parameters of statistical distributions. Choosing appropriate prior distributions for the parameter space is crucial for reducing the error of the Bayesian estimator. Therefore, it is important to define an appropriate prior distribution and set specific conditions for the hyperparameters.

Examples of such estimators are E-Bayes and hierarchical Bayes estimates. The hierarchical Bayes prior distribution was first introduced by Lindley and Smith (1972) and further studied by Han (1997), who also introduced the E-Bayes and hierarchical Bayes estimation methods. These methods have been utilized in estimation theory by several authors. For instance, Han (2009) used them to estimate the parameter of the exponential distribution and the ratio of the binomial distribution. Jaheen and Okasha (2011) used them to estimate the parameter and reliability function of the Burr Type XII distribution based on Type II right-censored samples, and Wang *et al.* (2012) used them to estimate the parameter of the Pascal distribution. Several authors have demonstrated the application of the hierarchical Bayes method in data analysis, including Micheas and Wikle (2009), Cressie and Tingley (2010), Ando and Zellner (2010), Osei *et al.* (2011) and Morey (2011). Recently, Makhdoom *et al.* (2023) obtained the E-Bayesian and hierarchical Bayesian estimation of reliability in multicomponent stress-strength models based on the inverse Rayleigh distribution.

Fuzzy sets have been utilized in estimation theory by several authors. Coppi *et al.* (2006) discussed various applications of Bayesian methods in statistical analysis. Huang *et al.* (2006) proposed a novel approach for determining the membership function of parameter estimates and the reliability function of multi-parameter lifetime distributions. Akbari and Rezaei (2007) introduced a fresh method for fuzzy point estimation for uniformly minimum variance. Pak *et al.* (2013) conducted extensive studies on statistical inference methods for lifetime distributions based on fuzzy numbers. Yaghoobzadeh Shahrastani (2019) derived E-Bayes and hierarchical Bayes estimates of the scalar parameter of the Gompertz distribution based on fuzzy data. Recently, Makhdoom and Pak (2024) applied a Bayesian approach in the Burr-type XII model based on fuzzy data.

The field of queueing theory has received significant attention from researchers, particularly with regard to maximum likelihood and Bayesian estimation methods. For example, Clarke (1957) focused on using maximum likelihood estimation to determine parameters in the steady state for the $M/M/1$ queueing model. Muddapur (1972) explored Bayesian estimation of arrival and service rates in both $M/M/1$ and $M/M/\infty$ queueing models. Thiruvaiyaru and Basawa (1992) investigated empirical Bayesian estimation of parameters in $M/M/1$ and $M/M/\infty$ queueing models, along with their asymptotic properties. Chowdhury and Mukherjee (2011) examined the estimation of waiting time in the right-skewed distribution of the $M/M/1$ queueing model. Chowdhury and Mukherjee (2013) provided maximum likelihood and Bayesian estimates of the traffic intensity parameter in the $M/M/1$ queueing model. Singh and Acharya (2019) compared the results of parameter estimation in the $M/M/1$ queueing model using Bayesian and maximum likelihood methods. Also, Goldenshluger and Koops (2019) focused on nonparametric estimation of service time in queueing models with infinite servers and Poisson input. Schweer and Wichelhaus (2020) explored nonparametric estimation of service time distribution in discrete-time queueing models in 2020. Lastly, Chandrasekhar *et al.* (2021) presented maximum likelihood and Bayesian estimates of the traffic intensity parameter in the $M/D/1$ queueing model.

Queueing systems with finite capacity are also highly significant and have numerous practical applications. Balsamo *et al.* (2003) studied network queueing systems to estimate the performance of software architectures. Takagi *et al.* (2005) addressed capacity calculations in wireless systems. Bocharov and Viskova (2005), Jain (2005), Gupta and Sikdar (2006), and Thomas (2006) conducted applied research on queueing systems with finite capacity. Factors such as system cost and customer satisfaction are considered evaluation criteria for a queuing system. Efforts are made to minimize system cost and maximize customer satisfaction in each queuing system.

The selection of the best estimate for the traffic intensity parameter of a queuing model based on a combined criterion of fuzzy and non-fuzzy indices, incorporating a factor called the average customer satisfaction level in choosing the appropriate estimate and the method for selecting the optimal estimate, is considered an innovation of this paper. No study has compared estimators using this approach and these criteria; in this article, it has been applied for the first time, and this is the main motivation of the study. This article focuses on obtaining shrinkage-based maximum likelihood estimation, as well as Bayesian, E-Bayesian, and hierarchical Bayesian estimations of the traffic intensity parameter under the general entropy loss function. The goal is to select an estimation that minimizes system cost and maximizes the average customer satisfaction level. The selection of the best estimate of the traffic intensity parameter for a queuing model based on a combined criterion of fuzzy and non-fuzzy indicators, incorporating a factor called the average customer satisfaction level in selecting the appropriate estimate and method of selecting the desired estimate in the article, is considered as an innovation of the article. To achieve this goal, the article introduces a decision criterion called $ACSI$. $ACSI$ is a linear weighted combination of system cost and average customer satisfaction used to determine the desired parameter estimator for traffic intensity. The method of selecting this desired estimator based on $ACSI$ is explained in the fourth part of the article, using Monte Carlo simulation and a numerical example.

The structure of the article is as follows: Section 2 defines E-Bayesian and hierarchical Bayesian estimations and introduces the queuing model $M/M/1/K$

along with its cost. In Section 3, we obtain the shrinkage-based maximum likelihood estimation, Bayesian, E-Bayesian, and hierarchical Bayesian estimations of the traffic intensity parameter for the introduced queuing model. Section 4 introduces the shrinkage estimation of ρ. Section 5 introduces the decision criterion ACSI and compares the obtained parameter estimations for traffic intensity using Monte Carlo simulation and two numerical examples, and finally, Section 6 presents the results of the article.

2. Preliminary definitions and concepts

In this section, we define the E-Bayes estimate, hierarchical Bayes estimate, fuzzy set probability function, and customer satisfaction degree average. We then introduce the $M/M/1/K$ queuing model, its evaluation criteria, and cost function. According to Han (1997), the definitions of E-Bayes and hierarchical Bayes estimates are as follows:

Definition 2.1. *Suppose b_1 and b_2 are hyperparameters in the prior distribution of θ, and $\pi(b_1, b_2)$ is the joint prior distribution of (b_1, b_2). Let $\hat{\theta}B(b_1, b_2)$ be the Bayesian estimator of θ. Then, the E-Bayesian estimate of θ, denoted by $\hat{\theta}EB$, is given by:*

$$\begin{aligned} \hat{\theta}EB &= E\pi(b_1, b_2)(\hat{\theta}B(b_1, b_2)) \\ &= \int \Lambda_1 \int_{\Lambda_2} \hat{\theta}_B(b_1, b_2)\pi(b_1, b_2)db_1 db_2, \quad b_1 \in \Lambda_1, ; b_2 \in \Lambda_2 \end{aligned}$$

This is the mathematical expectation of the Bayesian estimator of θ.

Definition 2.2. *If $\pi(\theta|\lambda)$ and $\pi^{'}(\lambda)$ are the prior distributions corresponding to the parameter θ and hyperparameter λ, respectively, then the hierarchical prior distribution of θ is obtained as:*

$$\pi^{''}(\theta) = \int_{\Lambda} \pi(\theta|\lambda)\pi^{'}(\lambda)d\lambda, \quad \lambda \in \Lambda,$$

If (Ω, F, P) is a probability space, where Ω is the sample space, F is a sigma algebra on Ω, and P is a probability measure, then the fuzzy set $\widetilde{A}$ in Ω is called a fuzzy event.

Definition 2.3. *(Zadeh (1968)): If, for all $\omega \in \Omega$, $\mu_{\widetilde{A}}(\omega)$ is the membership function of the fuzzy event A, then the probability function of A is defined as:*

$$P(\widetilde{A}) = \sum_{\omega \in \Omega} \mu_{\widetilde{A}}(\omega) P_{\omega}, \quad \mu_{\widetilde{A}}(\omega) : \Omega \rightarrow [0, 1]$$

2.1 Customer satisfaction degree average

This article aims to calculate the degree of customer satisfaction for the $M/M/m/K$ model by observing the queue length upon entry into the system, as per the method presented by Pardo and De la Fuente (2008). Based on whether customers encounter short, medium, or long queues, their satisfaction level will be classified as high (a_1), medium (a_2), or low (a_3), respectively, where ($a_1 \geq a_2 \geq a_3$).

The queues are represented as fuzzy sets: ($\widetilde{A}$: short queue), ($\widetilde{B}$: medium queue), and ($\widetilde{C}$: long queue). These can be defined as:

$$\begin{aligned}
\widetilde{A} &= (0, \mu_{\widetilde{A}}(0)), (1, \mu_{\widetilde{A}}(1)), \cdots, (K, \mu_{\widetilde{A}}(K)) \\
\widetilde{B} &= (0, \mu_{\widetilde{B}}(0)), (1, \mu_{\widetilde{B}}(1)), \cdots, (K, \mu_{\widetilde{B}}(K)) \\
\widetilde{C} &= (0, \mu_{\widetilde{C}}(0)), (1, \mu_{\widetilde{C}}(1)), \cdots, (K, \mu_{\widetilde{C}}(K)),
\end{aligned}$$

where $\mu_{\widetilde{A}}$, $\mu_{\widetilde{B}}$, and $\mu_{\widetilde{C}}$ are the membership functions of the fuzzy sets $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, respectively. According to Dubois (1980), the sum of the membership functions at each point equals 1:

$$\mu_{\widetilde{A}}(i) + \mu_{\widetilde{B}}(i) + \mu_{\widetilde{C}}(i) = 1, \quad i = 1, 2, \cdots, K.$$

Also, $\mu_{\widetilde{A}}(i)$ represents the degree of membership of the fuzzy set $\widetilde{A}$ when there are i customers in the queue. Therefore, according to definition 2.3, the probability of a customer entering the system encountering queues of short, medium and long lengths is

$$\begin{aligned}
\pi(\widetilde{A}) &= \sum_{n=0}^{K} \mu_{\widetilde{A}}(n) P_n \\
\pi(\widetilde{B}) &= \sum_{n=0}^{K} \mu_{\widetilde{B}}(n) P_n \\
\pi(\widetilde{C}) &= \sum_{n=0}^{K} \mu_{\widetilde{C}}(n) P_n. \qquad (2.2)
\end{aligned}$$

As a result, the average degree of customer satisfaction is defined as:

$$ADCS = a_1 \pi(\widetilde{A}) + a_2 \pi(\widetilde{B}) + a_3 \pi(\widetilde{C}) \qquad (2.3)$$

2.2 $M/M/1/K$ queuing model

The $M/M/1/K$ queuing system has one server with a service rate of μ, independent of the number of customers in the system. The arrival rate is λ, independent of the system's status. In this model, the time between arrivals and services is exponentially distributed with parameters λ and μ. When the system is full, the exit rate is different from the service rate.

In any queuing system, when the number of customers in the system at a specific moment is n, the time it takes for the system population to reach $n+1$ is a random variable with an exponential distribution with parameter λ_n. Similarly, when the system population is at n, the time to reach $n-1$ follows an exponential distribution with parameter μ_n. In the $M/M/1/K$ queuing model, we have:

$$\lambda_n = \begin{cases} \lambda & n = 0, 1, \cdots, K-1 \\ 0 & n \geq k \end{cases}$$

and

$$\mu_n = \begin{cases} \mu & n = 0, 1, \cdots, K \\ 0 & n > k. \end{cases}$$

In this queuing model, $\rho = \frac{\lambda}{\mu}$ is called the traffic intensity parameter. According to Allen (1990), the distribution of the number of customers in the system is:

$$P_n = \begin{cases} \frac{\rho^n(1-\rho)}{1-\rho^{K+1}} & n = 0, 1, \cdots, K, \quad \rho \neq 1 \\ \frac{1}{K+1} & n = 0, 1, \cdots, K, \quad \rho = 1 \end{cases} \tag{2.4}$$

The average number of customers in the system is

$$L_s = \begin{cases} \frac{\rho[1-(K+1)\rho^K + K\rho^{K+1}]}{(1-\rho)(1-\rho^{K+1})} & \rho \neq 1 \\ \frac{K}{2} & \rho = 1 \end{cases} \tag{2.5}$$

and the average number of customers in the queue, denoted $L_q = \sum_{n=1}^{K}(n-1)P_n$, is derived from $L_q = L_s - (1 - P_0)$, where:

$$P_0 = \begin{cases} \frac{1-\rho}{1-\rho^{K+1}} & \rho \neq 1 \\ \frac{1}{K+1} & \rho = 1 \end{cases} \tag{2.6}$$

The average waiting times for each customer in the queue (W_q) and in the system (W_s) are:

$$W_q = \frac{L_q}{\lambda(1-P_K)}, \quad W_s = \frac{L_s}{\lambda(1-P_K)} \tag{2.7}$$

The entry rate into the system is $\bar{\lambda} = \lambda(1 - P_K)$. If a customer cannot enter the system due to full capacity, we consider that customer as "lost" in this study.

2.3 The cost function for $M/M/1/K$ queuing model

In any queuing system, the goal is to reduce queue length and customer waiting time while increasing satisfaction, often by increasing the number of servers. However, this increase incurs additional costs. In general, the expected total cost per unit time serves as a key evaluation criterion and depends on the specific type and nature of the system. In this study, the cost function is proposed as:

$$C(\rho) = C_1(\lambda - \bar{\lambda}) + C_2 L_q + C_3(L_s - L_q) = C_1 \lambda P_K + (C_2 - C_3)L_q + C_3 L_s \quad (2.8)$$

where:

- $C_1(\lambda - \bar{\lambda})$ is the cost of lost customers, incurred when the system prevents entry due to full capacity or when customers choose not to enter due to congestion. Since λ is the arrival rate and $\bar{\lambda}$ is the actual entry rate, $\lambda - \bar{\lambda}$ represents the average number of customers rejected by the system. Therefore, if each lost customer has a cost C_1, the average loss due to lost customers per unit time is $C_1(\lambda - \bar{\lambda})$.
- $C_2 L_q$ represents the cost of customers' time wasted in the queue, with a total cost equal to the time cost per customer (C_2) multiplied by the average number of customers in the queue (L_q).
- $C_3(L_s - L_q)$ is the cost of customers' time while receiving service, with the total average cost equal to the time cost per customer (C_3) times the average number of customers receiving service, ($L_s - L_q$).

An essential consideration in this article is that the $M/M/1/K$ queuing model is evaluated for cases where $\rho \neq 1$.

3. Estimation of the Traffic Intensity Parameter (ρ)

In this section, we derive shrinkage-based estimates for the traffic intensity parameter (ρ) using maximum likelihood estimation, Bayesian estimation, E-Bayesian, and hierarchical Bayesian methods under the general entropy loss function:

$$L(\hat{\theta}, \theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^p - p \ln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \quad p \neq 0.$$

Generally, a Bayesian estimate of a parameter θ under the general entropy loss function (see Dey *et al.* (1986)) is given by

$$\hat{\theta}^B = [E(\theta^{-p}|\mathbf{X})]^{-\frac{1}{p}}. \tag{3.9}$$

Assume $V_1, \cdots, V_{n_1}$ are independent random variables representing times between consecutive entries, following an exponential distribution with parameter λ, with probability density function $f(\nu, \lambda)$ given by

$$f(v, \lambda) = \lambda e^{-\lambda v}, \quad v > 0, \lambda > 0.$$

Similarly, let $U_1, \cdots, U_{n_2}$ represent service times, which are independent random variables following an exponential distribution with parameter μ and probability density function $g(u, \mu)$:

$$g(u, \mu) = \mu e^{-\mu u}, \quad u > 0, \mu > 0.$$

Here, λ and μ are independent parameters, with λ following a $\Gamma(a, b)$ distribution with probability density function

$$\pi(\lambda|a, b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}, \quad \lambda > 0, a > 0, b > 0, \tag{3.10}$$

and μ following a $\Gamma(r, c)$ distribution with probability density function

$$\pi(\mu|r, c) = \frac{c^r}{\Gamma(r)} \mu^{r-1} e^{-c\mu}, \quad \mu > 0, r > 0, c > 0. \tag{3.11}$$

4. Shrinkage Estimation

Let $T_1 = \sum_{i=1}^{n_1} V_i$ and $T_2 = \sum_{i=1}^{n_2} U_i$. The MLE of ρ is given by

$$\hat{\rho}_M = \frac{\hat{\lambda}}{\hat{\mu}} = \frac{n_1}{n_2} \frac{T_2}{T_1}.$$

Using Eq.(1.1) and letting $F = \frac{T_2}{T_1}$, the shrinkage estimate of ρ becomes

$$\hat{\rho}_T = m\hat{\rho}_M + (1-m)\rho_0 = \frac{mn_1}{n_2} F + (1-m)\rho_0. \tag{4.12}$$

Given that T_1 and T_2 follow $Gamma(n_1, \frac{1}{\lambda})$ and $Gamma(n_2, \frac{1}{\mu})$ distributions, respectively, the density function of $F = \frac{T_2}{T_1}$ is

$$g_F(f) = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \frac{f^{n_2-1}\rho^{n_1}}{(f+\rho)^{n_1+n_2}}, \quad f > 0.$$

The risk function of the shrinkage estimate of ρ is

$$\begin{aligned} R(\rho) =& \frac{1}{\rho^p} \int_0^\infty \left(\frac{mn_1}{n_2} f + (1-m)\rho_0 \right)^p g(f) df \\ & - p \int_0^\infty \ln \left(\frac{mn_1}{n_2} f + (1-m)\rho_0 \right) g(f) df + p \ln \rho - 1. \end{aligned}$$

To minimize $R(\rho)$, the optimal m is obtained by setting $\frac{dR(\rho)}{dm} = 0$:

$$\int_0^\infty \left(\frac{n_1}{n_2} f - \rho_0 \right) \left(\frac{A^{p-1}}{\rho^p} - \frac{1}{A} \right) g(f) df = 0,$$

where $A = \frac{mn_1}{n_2} f + (1-m)\rho_0$. Under the condition $\frac{n_1}{n_2} f \neq \rho_0$ and assuming $\delta = \rho - \rho_0$, the optimal m is obtained as

$$m_{Opt} = \frac{\delta}{\frac{n_1}{n_2} f - \rho_0}.$$

Under the condition $\rho_0 < \rho < \frac{n_1}{n_2} f$, we obtain $0 < m_{Opt} < 1$. Thus, Eq.(4.12) is rewritten as

$$\hat{\rho}T = \frac{mOptn_1}{n_2} F + (1 - m_{Opt})\rho_0. \tag{4.13}$$

4.1 Bayesian estimation method

Assuming $\mathbf{X} = U_1, \cdots, U_{n_2}, V_1, \cdots, V_{n_1}$ and following Eqs. (3.10) and (3.11), we obtain:

$$\pi(\lambda, \mu|\mathbf{X}) = \frac{\Gamma(n_1+a)\Gamma(n_2+r)}{(T_1+b)^{T_1+a}(T_2+c)^{T_2+r}} \lambda^{n_1+a-1} \mu^{n_2+r-1} e^{-\lambda(T_1+b)-\mu(T_2+c)}. \tag{4.14}$$

Assuming $\phi = \frac{\Gamma(a+n_1-p)\Gamma(r+n_2+p)}{\Gamma(a+n_1)\Gamma(r+n_2)}$, the Bayesian estimate of ρ under the general entropy loss function, using Eqs. (3.9) and (4.14), is:

$$\hat{\rho}_B(b,c) = E(\rho^{-p}|\mathbf{X})^{-\frac{1}{p}} = (\phi(T))^{-\frac{1}{p}} . \frac{c+T_2}{b+T_1}. \tag{4.15}$$

4.2 E-Bayesian estimation method

According to Han (1997), in Eq. (3.10), a and b are chosen so that $\pi(\lambda|a,b)$ is decreasing with respect to λ. Consequently, to satisfy:

$$\frac{d\pi(\lambda|a,b)}{d\lambda} = \frac{b^a\lambda^{a-2}e^{b\lambda}}{\Gamma(a)}\left((a-1)-b\lambda\right),$$

it must be that $b>0$ and $0<a\leq 1$. Berger (2013) showed that as b increases, the Bayesian estimate of λ becomes less robust; thus, the hyperparameter b is bounded by $0<b<c_1$, where c_1 is a constant. Following Han (2011), b is best modeled with a uniform distribution over $(0,c_1)$. Assuming $a=1$, Eq. (3.10) simplifies to:

$$\pi(\lambda|b) = be^{-b\lambda}, \quad \lambda>0, b>0. \tag{4.16}$$

Similarly, for r and c in Eq. (3.11), c is uniformly distributed over $(0,c_2)$, where c_2 is a constant. Assuming $r=1$, Eq. (3.11) simplifies to:

$$\pi(\mu|c) = ce^{-c\mu}, \quad ,\mu>0, c>0. \tag{4.17}$$

Using Definition 1 and Eqs. (4.15) to (4.17), the E-Bayesian estimate of ρ is:

$$\begin{aligned}\hat{\rho}_{EB} &= \frac{1}{c_1c_2}\int_0^{c_2}\int_0^{c_1}\hat{\rho}_B(b,c)\pi(b,c)dbdc\\ &= \frac{(\phi(T))^{-\frac{1}{p}}\left[(c_2+T_1)^2-T_1^2\right]}{2c_1c_2}\log\left(\frac{c_1+T_2}{T_2}\right)\end{aligned} \tag{4.18}$$

4.3 Hierarchical Bayesian estimate

According to Eqs. (4.16) and (4.17) and using Definition 2, the hierarchical prior distributions of the parameters λ and μ are respectively obtained as

$$\pi(\lambda) = \int_0^{c_1}\pi(\lambda|b)\pi(b)db = \frac{1-(1+c_1\lambda)e^{-c_1\lambda}}{c_1\lambda^2} \tag{4.19}$$

and

$$\pi(\mu) = \int_0^{c_2}\pi(\mu|c)\pi(c)dc = \frac{1-(1+c_2\mu)e^{-c_2\mu}}{c_2\mu^2}. \tag{4.20}$$

Therefore, according to Eqs. (4.19) and (4.20), the hierarchical posterior distribution of λ and μ is

$$\pi^{**}(\lambda,\mu|\mathbf{X}) = \frac{\lambda^{a+n_1-3}\mu^{r+n_2-3}e^{-\lambda(b+T_1)-\mu(c+T_2)}S(\lambda,\mu)}{\int_0^\infty\int_0^\infty\lambda^{a+n_1-3}\mu^{r+n_2-3}e^{-\lambda(b+T_1)-\mu(c+T_2)}S(\lambda,\mu)d\lambda d\mu} \tag{4.21}$$

in which

$$S(\lambda,\mu)=\left(1-c_1\lambda e^{-c_1\lambda}-e^{-c_1\lambda}\right)\left(1-c_2\mu e^{-c_2\mu}-e^{-c_2\mu}\right).$$

Using Eq. (4.21), the hierarchical Bayesian estimate of ρ under the general entropy loss function is as

$$\hat{\rho}_{HB}=\left\{\frac{\int_0^\infty\int_0^\infty \lambda^{n_1-(p+2)}\mu^{n_2+p-2}e^{-\lambda T_1-\mu T_2}S(\lambda,\mu)d\lambda d\mu}{\int_0^\infty\int_0^\infty \lambda^{n_1-2}\mu^{n_2-2}e^{-\lambda T_1-\mu T_2}S(\lambda,\mu)d\lambda d\mu}\right\}^{-\frac{1}{p}} \tag{4.22}$$

and is obtained using the Lindley approximation method. To calculate $\hat{\rho}_{HB}$ in Eq. (4.22), we can also use the Lindley approximation (Lindley (1980)) as follows:

In general, the result of the integral ratio of the form

$$E(u(\Lambda)|\mathbf{X})=\frac{\int u(\Lambda)e^{Q(\Lambda)}d\Lambda}{\int e^{Q(\Lambda)}d\Lambda} \tag{4.23}$$

in which $Q(\Lambda)=l(\Lambda)+\rho(\Lambda)$, in a way that $l(\Lambda)$ is the logarithm of the likelihood function of observations and $\rho(\Lambda)$ is the logarithm of the prior distribution of Λ, the Lindley approximation method is used to obtain the result as follows:

$$E(u(\Lambda)|\mathbf{X})=\{u+\frac{1}{2}\sum_i\sum_j(u_{ij}+2u_i\rho_j)\sigma_{ij}+\frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l\}_{\hat{\Lambda}} \tag{4.24}$$

in which $\Lambda=(\lambda_1,\cdots,\lambda_m)$, $i,j,k,l=1,\cdots,m$ and $\hat{\Lambda}$ is the MLE of Λ. We also have:

$$u=u(\Lambda),u_i=\frac{\partial u}{\partial\lambda_i},u_{ij}=\frac{\partial^2 u}{\partial\lambda_i\partial\lambda_j},L_{ijk}=\frac{\partial^3 l(\Lambda)}{\partial\lambda_i\partial\lambda_j\partial\lambda_k},\rho_j=\frac{\partial\rho(\Lambda)}{\partial\lambda_j}$$

where σ_{ij} is the element (i,j) of the inverse of the matrix $\{-L_{ij}\}$. For the case of $\Lambda=(\lambda_1,\lambda_2)$, Eq. (3.11) is obtained as follows:

$$E(u(\Lambda)|\mathbf{X})=(u+Au_1+Bu_2+C)_{\hat{\lambda}},\quad \hat{\Lambda}=(\hat{\lambda}_1,\hat{\lambda}_2)$$

in which

$$A=\rho_1\sigma_{11}+\rho_2\sigma_{12}+\frac{1}{2}\sum_{i=1}^2\sum_{j=1}^2 L_{ij}\sigma_{ij},\quad B=\rho_1\sigma_{21}+\rho_2\sigma_{22}+\frac{1}{2}\sum_{i=1}^2\sum_{j=1}^2 L_{ij}\sigma_{ij}$$

$$C=\frac{1}{2}\sum_{i=1}^2\sum_{j=1}^2 u_{ij}\sigma_{ij}.$$

Therefore, assuming $\Lambda=(\lambda,\mu)$, $u(\lambda,\mu)=(\lambda\mu)^{-p}$ and,

$$l(\Lambda)=n_1\ln\lambda-\lambda T_1+n_2\ln\mu-\mu T_2,\quad \rho(\Lambda)\propto\ln S(\lambda,\mu)-2\ln\lambda-2\ln\mu$$

we have:

$$\rho_1 = \frac{(c_1\lambda^2 + 2c_1\lambda - 2)e^{-c_1\lambda} - 2}{\lambda[1-(1+c_1\lambda)e^{-c_1\lambda}]}, \quad \rho_2 = \frac{(c_2\mu^2 + 2c_2\mu - 2)e^{-c_2\mu} - 2}{\mu[1-(1+c_2\mu)e^{-c_2\mu}]}$$

$$L_{11} = -\frac{n_1}{\lambda^2}, \quad L_{12} = L_{21} = 0, \quad L_{22} = -\frac{n_2}{\mu^2}$$

$$u_1 = -\frac{p}{\lambda(\lambda\mu)^p}, \quad u_2 = -\frac{p}{\mu(\lambda\mu)^p}, \qquad u_{12} = u_{21} = \frac{p^2}{(\lambda\mu)^{p+1}}$$

$$u_{11} = \frac{p(p+1)}{\lambda^{p+2}\mu^{2p}}, \quad u_{22} = \frac{p(p+1)}{\mu^{p+2}\lambda^{2p}}$$

$$\sigma_{11} = \frac{\lambda^2}{n_1}, \quad \sigma_{12} = \sigma_{21} = 0, \quad \sigma_{22} = \frac{\mu^2}{n_2}$$

$$A = \frac{\lambda\rho_1 - n_1^2}{n_1^2}, \quad B = \frac{\mu\rho_2 - n_2^2}{n_2^2}$$

$$C = p(p+1)\left[\frac{1}{n_1(\lambda\mu^2)^p} + \frac{1}{n_2(\lambda^2\mu)^p}\right].$$

Therefore, $\hat{\rho}_{HB}$ is obtained using the Lindley approximation method as follows:

$$\hat{\rho}_{HB} = \hat{\lambda}\hat{\mu}\left[1 - \frac{p(\hat{\lambda}\hat{\rho}_1 - n_1^2)}{\hat{\lambda}n_1^2} - \frac{p(\hat{\mu}\hat{\rho}_2 - n_2^2)}{\hat{\mu}n_2^2} + p(p+1)\hat{\mu}^{-p} + p(p+1)\hat{\lambda}^{-p}\right]^{-\frac{1}{p}} \quad (4.25)$$

where $\hat{\lambda}$ and $\hat{\mu}$ are the MLEs of λ and μ, respectively, and they are given by:

$$\hat{\lambda} = \frac{n_1}{T_1}, \quad \hat{\mu} = \frac{n_2}{T_2}$$

5. Simulation and data analysis

In this section, we consider the $M/M/1/8$ queuing model and obtain the shrinkage, Bayesian, E-Bayesian, and hierarchical Bayesian estimates of ρ. We then calculate the cost function values for these estimates via the Monte Carlo sample mean simulation method and determine the average degree of customer satisfaction based on the estimates of ρ. The simulation process includes the following steps:

- Step 1: Generate a sample of size $n_1 = 35$ $(V_1, \cdots, V_{n_1})$ from an exponential distribution with parameter $\lambda = 4$, and a random sample of size $n_2 = 30$ $(U_1, \cdots, U_{n_2})$ from an exponential distribution with parameter $\mu = 6$.

- Step 2: Assuming $\rho_0 = 0.5$ and $\delta = 0.4$, calculate the shrinkage estimate of ρ from Eq. (4.13) with $b = 4$, $a = 3$, $r = 3$, $c = 5$, and $p = -2.5$. Calculate the Bayesian estimate of ρ from Eq. (4.15), and, assuming $c_1 = 6$ and $c_2 = 7$, obtain the E-Bayesian and hierarchical Bayesian estimates of ρ from Eqs. (4.18) and (4.22). Repeat steps 1 to 3 for 5000 iterations and use the average of these estimates as the final values.

- Step 3: For $C_1 = 250$, $C_2 = 200$, and $C_3 = 150$, calculate the cost function for each estimate of ρ from Eq. (2.8). The simulation results are summarized in Table 1.

Next, assuming $a_2 = 0.6$, $a_1 = 1$, and $a_3 = 0.35$ and defining fuzzy sets $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ as:

$$\widetilde{A} = \{(0, 0.6), (1, 0.5), (2, 0.7), (3, 0.8), (4, 0.9), (5, 0.2), (6, 0), (7, 0.4), (8, 0.8)\}$$
$$\widetilde{B} = \{(0, 0.3), (1, 0.3), (2, 0.2), (3, 0.1), (4, 0), (5, 0.7), (6, 1), (7, 0.5), (8, 0.1)\}$$
$$\widetilde{C} = \{(0, 0.1), (1, 0.2), (2, 0.1), (3, 0.1), (4, 0.1), (5, 0.1), (6, 0), (7, 0.1), (8, 0.1)\}.$$

We calculate the average degree of customer satisfaction for the $M/M/1/8$ model for different estimates of ρ described in this article. Prior to this, we obtain the distribution of the number of customers present in the system for each estimate of ρ, with results shown in Table 2.

Using Eq. (2.2) and Table 2, we calculate the probability of each fuzzy set $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, and then, via Eq. (2.3), compute the average degree of customer satisfaction under shrinkage, Bayesian, E-Bayesian, and hierarchical Bayesian estimates of ρ. Results are recorded in Table 3.

Table 1: Estimates of ρ, L_s, L_q and $C(\rho)$

	$\hat{\rho}_T$	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$
Estimates	0.59874	1.0226	0.96972	1.0749
L_s	1.4312	4.6842	4.2467	5.0907
L_q	0.83509	3.7745	3.3611	4.1614
$\bar{\lambda}$	3.9841	3.5586	3.6530	3.4581
$C(\hat{\rho})$	260.41	1001.7	991.81	1107.2

Subsequently, the average customer satisfaction values calculated through shrinkage estimation methods—Bayesian, Empirical Bayesian, and hierarchical Bayesian are normalized. This normalization process involves dividing each value by the highest value among them so that the estimator with the highest average customer satisfaction attains a normalized score of 1. Similarly, for the cost function values calculated under these estimation methods, the normalization is performed by dividing the smallest cost value by each of the values, assigning the highest normalized score of 1 to the estimate with the lowest cost.

Both criteria are thus normalized to ensure that lower system costs and higher customer satisfaction yield higher normalized values. To determine a suitable estimator, an index based on the normalized system cost (C_N) and normalized

Table 2: The distribution of the number of customers in the system in the $M/M/1/8$ model for different estimates of ρ.

	$\hat{\rho}_T$	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$
P_0	0.40527	0.10114	0.12526	0.08180
P_1	0.24265	0.10373	0.12146	0.08793
P_2	0.14528	0.10608	0.11779	0.09451
P_3	0.08699	0.10847	0.11422	0.10159
P_4	0.05208	0.11093	0.11076	0.10920
P_5	0.03118	0.11343	0.10741	0.11738
P_6	0.01867	0.11599	0.10416	0.12617
P_7	0.01118	0.11862	0.10100	0.13562
P_8	0.00669	0.12129	0.09794	0.14578

Table 3: The probability of fuzzy sets and the value of $ADCS$ under the methods of estimating ρ

	$\hat{\rho}_T$	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$
$\pi(\widetilde{A})$	0.59847	0.54058	0.54801	0.53310
$\pi(\widetilde{B})$	0.27886	0.36035	0.34783	0.37070
$\pi(\widetilde{C})$	0.12237	0.09874	0.10146	0.09617
$ADCS$	0.80862	0.79135	0.79222	0.78918

average customer satisfaction (A_N) is introduced, defined as:

$$ACSI = w_1 A_N + w_2 C_N, \tag{5.26}$$

where $w_1 + w_2 = 1$. This ACSI index is directly related to both normalized criteria, implying that higher customer satisfaction and lower system costs result in a higher ACSI index value. Therefore, an estimator with a larger ACSI index is considered more suitable.

Assuming $w_1 = 0.6$ and $w_2 = 0.4$,the ACSI index is calculated for the normalized values of average customer satisfaction and cost function. The normalized results and the calculated ACSI index values for the shrinkage, Bayesian, Empirical Bayesian, and hierarchical Bayesian estimators are summarized in Table 4. According to the ACSI index values in Table 4, the shrinkage estimate of ρ is preferable over the other estimates. Among the Bayesian estimates, the Empirical Bayesian estimate performs better than both the Bayesian and hierarchical Bayesian estimates.

Table 4: The results of normalizing the average customer satisfaction values and cost function

	$\hat{\rho}_T$	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$
C_N	1	0.25997	0.26256	0.23519
A_N	1	0.97864	0.97972	0.97596
$ACSI$	1	0.54744	0.54942	0.53149

5.1 Numerical Example 2

Consider an insurance company with a single employee dedicated to processing car accident claim payments in a designated room on a specific day. The room has a limited capacity, accommodating up to 10 individuals at any given time. The time intervals (in minutes) between successive customer arrivals and the service times are recorded in Table 5. Based on the data and hypothetical values previously

Table 5: Corresponding data for the insurance company

Time intervals between inputs	3.98	2.35	2.91	2.67	2.27	4.32	3.15	5.40
	4.71	3.46	4.84	6.52	2.12	3.08	4.34	4.11
	4.17	2.68	3.82	3.50	3.32	3.97	4.95	5.00
	3.68	4.38	2.52	1.77	2.37	3.25		
Time intervals between services	3.44	0.73	1.50	0.63	0.11	0.64	3.86	2.62
	0.17	1.51	2.80	1.97	5.17	0.56	1.90	2.15
	0.49	10.00	3.15	4.47	2.93	0.34	0.94	0.18
	1.86	4.25	0.10	2.94	3.15	2.19		

assumed for parameters ρ_0, δ, a, b, c, r, p, c_1, c_2, C_1, C_2, and C_3, various estimates for ρ and other relevant evaluation criteria have been calculated and recorded in Table 6. Additionally, the distribution of customers currently in the system for the insurance company model has been obtained and is recorded in Table 7.

Table 6: Estimates ρ, L_s, L_q and $C(\rho)$

	$\hat{\rho}_T$	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$
Estimates	1.5783	2.2936	2.4821	2.3505
L_s	8.3439	9.2282	9.3258	9.2604
L_q	7.3478	8.2283	8.3258	8.2606
$\bar{\lambda}$	3.5976	2.4848	2.2963	2.4247
$C(\hat{\rho})$	2144.6	2599.4	2666.1	2620.9

Table 7: Distribution of the number of customers in the system for different estimates ρ

	$\hat{\rho}_T$	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$
P_0	0.00609	0.00032	0.00017	0.00026
P_1	0.00962	0.00074	0.00041	0.00062
P_2	0.01518	0.00169	0.00103	0.00145
P_3	0.02395	0.00388	0.00255	0.00341
P_4	0.03781	0.00889	0.00634	0.00801
P_5	0.05967	0.02039	0.01573	0.01883
P_6	0.09418	0.04676	0.03905	0.04426
P_7	0.14864	0.10724	0.09693	0.10402
P_8	0.23459	0.24596	0.24059	0.24449
P_9	0.37027	0.56414	0.59718	0.57467
P_{10}	0.58439	0.79239	0.84823	0.85067

Now assuming $a_1 = 1$, $a_2 = 0.7$ and $a_3 = 0.4$ and considering the fuzzy sets $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$

$$\widetilde{A} = \{(0, 0.6), (1, 0.5), (2, 0.7), (3, 0.8), (4, 0.9), (5, 0.2), (6, 0), (7, 0.4), (8, 0.8), (9, 0.4), (10, 0)\}$$

$$\widetilde{B} = \{(0, 0.3), (1, 0.3), (2, 0.2), (3, 0.1), (4, 0), (5, 0.7), (6, 1), (7, 0.5), (8, 0.1), (9, 0.5), (10, 0.2)\}$$

$$\widetilde{C} = \{(0, 0.1), (1, 0.2), (2, 0.1), (3, 0.1), (4, 0.1), (5, 0.1), (6, 0), (7, 0.1), (8, 0.1), (9, 0.1), (10, 0.8)\}$$

The probabilities of the fuzzy sets $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ are calculated using Eq.(2.2) and Table 7. Subsequently, the average customer satisfaction degree is obtained using Eq.(2.3) under shrinkage, Bayesian, E-Bayesian, and hierarchical Bayesian ρ estimates, with the results recorded in Table 8. Following the normalization method from the previous section, the cost function values and average customer satisfaction degrees for the insurance company example are normalized and presented in Table 9. Based on the ACSI index values in Table 9, it is concluded that in this example, the shrinkage estimator ρ is a more suitable estimator than the alternatives.

Table 8: Probability of fuzzy sets and values of $ADCS$ under the estimation methods of ρ

	$\hat{\rho}_T$	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$
$\pi(\widetilde{A})$	0.47945	0.48225	0.48203	0.48232
$\pi(\widetilde{B})$	0.54589	0.58084	0.59145	0.59235
$\pi(\widetilde{C})$	0.55906	0.72931	0.77471	0.77619
$ADCS$	1.0852	1.1806	1.2059	1.2074

Table 9: Results of normalizing the average customer satisfaction degree values, cost function values, and the ACSI index for the insurance company example

	$\hat{\rho}_T$	$\hat{\rho}_B$	$\hat{\rho}_{EB}$	$\hat{\rho}_{HB}$
C_N	1	0.82504	0.80439	0.81827
A_N	0.89789	0.97780	0.99876	1
$ACSI$	.93873	0.91669	0.92101	0.89096

6. Conclusion

In this article, the $M/M/1/K$ queueing model is considered, where the times between successive arrivals and service times follow exponential distributions with parameters λ and μ, respectively. The traffic intensity parameter of the model is estimated using shrinkage methods based on maximum likelihood estimation (MLE), Bayesian, E-Bayesian, and hierarchical Bayesian approaches under the general entropy loss function. This selection aims to minimize system cost while maximizing a criterion termed average customer satisfaction, as defined in the article.

An index called ACSI is introduced, which incorporates both system cost and average customer satisfaction, ensuring that an estimate with a larger ACSI index is regarded as more suitable for ρ. Finally, in the numerical analysis section, Monte Carlo simulation methods are employed for the $M/M/1/8$ model, alongside two numerical examples for the $M/M/1/10$ model. The findings indicate that the shrinkage estimate of ρ outperforms the other estimates. Additionally, among the Bayesian estimation methods discussed, the E-Bayesian estimate of ρ is determined to be the most appropriate.

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