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Inference on $Pr(X > Y)$ Based on Record Values From the Power Hazard Rate Distribution

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Abstract:

In this article, we consider the problem of estimating the stress-strength reliability $Pr(X > Y)$ based on upper record values when X and Y are two independent but not identically distributed random variables from the power hazard rate distribution with the common scale parameter k. When the parameter k is known, the maximum likelihood estimator (MLE), the approximate Bayes estimator, and the exact confidence intervals of stress-strength reliability are obtained. When the parameter k is unknown, we obtain the MLE and some bootstrap confidence intervals of the stress-strength reliability. We also apply the Gibbs sampling technique to study the Bayesian estimation of the stress-strength reliability and its corresponding credible interval. An example is presented to illustrate the inferences discussed in the previous sections. Finally, to investigate and compare the performance of the different proposed methods in this paper, a Monte Carlo simulation study is conducted.

Keywords: Bayes estimation, Maximum likelihood estimation, Monte Carlo simulation, Power hazard rate distribution, Record values, Stress-strength reliability.

Mathematics Subject Classification (2010): 62F10, 62F15, 62N05.

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1. Introduction

The problem of estimating $R = Pr(X > Y)$ arises in the context of mechanical reliability of a system with strength X and stress Y and R is chosen as a measure of system reliability. The system fails if and only if, at any time, the applied stress is greater than its strength. This type of reliability model is known as the stress-strength model (Valiollahi et al., 2013). This problem also arises in situations where X and Y represent lifetimes of two devices, and one wants to estimate the probability that one fails before the other. For example, in biometrical studies, the random variable X may represent the remaining lifetime of a patient treated with a certain drug while Y represents the remaining lifetime when treated by another drug (Al-Gashgari and Shawky, 2014). The estimation of stress-strength reliability is very common in statistical literature. For example, see Raqab and Kundu (2005), Kundu and Gupta (2005, 2006) and Nadar et al. (2014). Also, the reader is referred to the book by Kotz et al. (2003) for other applications and motivations for the study of the stress-strength reliability.

Record values arise naturally in many real-life applications involving data relating to meteorology, hydrology, sports, and life-tests. Record values also are of great importance to scientists and engineers and have been studied extensively. In industry and reliability studies, many products may fail under stress. For example, a wooden beam breaks when sufficient perpendicular force is applied to it, an electronic component ceases to function in an environment of too high temperature, and a battery dies under the stress of time (Soliman et al., 2006). But the precise breaking stress or failure point varies even among identical items. Hence, in such experiments, measurements may be made sequentially and only values larger (or smaller) than all previous ones are recorded (Soliman et al., 2006). Data of this type are called record data. The theory of record values was first introduced by Chandler (1952). For more details and applications of record values, readers may refer to the book by Arnold et al. (1998) and the references cited therein.

Let $\{X_i,\ i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with an absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Define $Y_m = \max\{X_1, ..., X_m\}$, $m \geq 1$. Then, X_j is an upper record value of the sequence $X_1, X_2, ...$, if $X_j > Y_{j-1}$, $j > 1$. Let $T_1 = 1$ with probability 1, and

$$T_n = \min\{j : j > T_{n-1}, X_j > X_{T_{n-1}}\}, \quad n > 1,$$

then $\{T_n, n \geq 1\}$ is the record time sequence, at which the records appear. Therefore, the sequence of upper record values is defined by $R_n = X_{T_n}$, $n \geq 1$. An analogous definition can be given for lower records.

The problem of estimating the stress-strength reliability based on record values for several distributions are discussed in the works of Baklizi (2014 a, b), Nadar and Kizilaslan (2014) and Al-Gashgari and Shawky (2014).

Mugdadi (2005) defined the power hazard function as

$$h(x) = ax^k, \quad x > 0, \quad a > 0, \quad k > -1.$$

Corresponding to this hazard function, the cumulative distribution function (cdf) is given by,

$$F(x) = 1 - \exp\left(-\frac{a}{k+1}x^{k+1}\right), \tag{1.1}$$

and the probability density function (pdf) is given by,

$$f(x) = ax^k \exp\left(-\frac{a}{k+1}x^{k+1}\right). \tag{1.2}$$

If X has pdf (1.2), we denoted it by $X \sim \text{PHRD}(a, k)$.

The power hazard function is very simple, and it can be increasing, decreasing, or constant. Therefore, the power hazard rate distribution (PHRD) can often provide a better fit than other two-parameter distributions when modeling monotone hazard rates.

Mugdadi and Min (2009) investigated the Bayes estimation for the power hazard rate distribution. Parameter estimation for the power hazard rate distribution based on record data is considered by Tarvirdizade and Nematollahi (2016). The problem of estimating the stress-strength reliability for the power hazard rate distribution is discussed by Kinaci (2014).

In this paper, we consider the problem of estimating the stress-strength reliability for the power hazard rate distribution based on upper record values. It is important to note that some well-known lifetime distributions such as exponential, Rayleigh, and Weibull are special cases of the PHRD defined in (1.2). Therefore, the results obtained in this paper can be valid for these distributions and the other distributions which have a power hazard function.

The rest of the paper is organized as follows. In Section 2, we discussed the likelihood inference for the stress-strength reliability. In Section 3, we presented some bootstrap confidence intervals for stress-strength reliability. In Section 4, Bayesian inference on R is considered. In Section 5, an example is presented to illustrate the inferences discussed in the previous sections. In Section 6, a Monte Carlo simulation study is conducted to investigate and compare the performance of different types of estimators presented in this paper. Finally, a discussions and some conclusion are given in Section 7.

2. Likelihood Inference

Let X and Y be independent random variables from the power hazard rate distribution with the parameters (a_1, k) and (a_2, k) respectively. Let $R = \Pr(X > Y)$ be the stress-strength reliability. Then, from (1.1) and (1.2) we have

$$\begin{aligned} R = \Pr(X > Y) &= \int_0^\infty \int_y^\infty f_X(x) f_Y(y) dx dy = \int_0^\infty f_Y(y)(1 - F_X(y)) dy \\ &= \int_0^\infty a_2 y^k e^{-\frac{a_2}{k+1} y^{k+1}} e^{-\frac{a_1}{k+1} y^{k+1}} dy = \frac{a_2}{a_1 + a_2}. \end{aligned}$$

Our interest is to estimate R based on upper record values on both variables. Let $\underset{\sim}{r} = (r_1, ..., r_n)$ be a set of upper records from PHRD(a_1, k) and let $\underset{\sim}{s} = (s_1, ..., s_m)$ be an independent set of upper records from PHRD(a_2, k). The likelihood functions are given by (Ahsanullah, 2004),

$$\begin{aligned} L(a_1, k|\underset{\sim}{r}) &= f(r_n) \prod_{i=1}^{n-1} \left(\frac{f(r_i)}{1 - F(r_i)} \right), \quad 0 < r_1 < ... < r_n < \infty, \\ L(a_2, k|\underset{\sim}{s}) &= g(s_m) \prod_{i=1}^{m-1} \left(\frac{g(s_i)}{1 - G(s_i)} \right), \quad 0 < s_1 < ... < s_m < \infty. \end{aligned} \tag{2.3}$$

where f and F are the pdf and cdf of $X \sim$ PHRD(a_1, k) respectively and g and G are the pdf and cdf of $Y \sim$ PHRD(a_2, k) respectively. Substituting f , F , g and G in (2.3), we obtain the likelihood functions as follows

$$\begin{aligned} L(a_1, k|\underset{\sim}{r}) &= a_1^n \exp\left(-\frac{a_1}{k+1} r_n^{k+1} \right) . \prod_{i=1}^n r_i^k, \\ L(a_2, k|\underset{\sim}{s}) &= a_2^m \exp\left(-\frac{a_2}{k+1} s_m^{k+1} \right) . \prod_{i=1}^m s_i^k. \end{aligned} \tag{2.4}$$

Hence, the joint Log-likelihood function of the observed records $\underset{\sim}{r}$ and $\underset{\sim}{s}$ is given by

$$\begin{aligned} \ell(a_1, a_2, k|\underset{\sim}{r}, \underset{\sim}{s}) = \; & n \ln a_1 + m \ln a_2 - \tfrac{a_1}{k+1} r_n^{k+1} - \tfrac{a_2}{k+1} s_m^{k+1} \\ & + k \left(\sum_{i=1}^n \ln r_i + \sum_{i=1}^m \ln s_i \right). \end{aligned} \tag{2.5}$$

Next, we consider likelihood inference for R in the following two cases:

2.1 When the parameter k is known

Under the assumption that the parameter k is known, the MLE of the parameters a_1 and a_2 based on the upper record values can be obtained by solving the following likelihood equations:

$$\frac{\partial \ell}{\partial a_1} = \frac{n}{a_1} - \frac{1}{k+1} r_n^{k+1} = 0, \qquad \frac{\partial \ell}{\partial a_2} = \frac{m}{a_2} - \frac{1}{k+1} s_m^{k+1} = 0. \tag{2.6}$$

Hence, the MLEs of a_1 and a_2, say $\hat{a}_1$ and $\hat{a}_2$, are given by

$$\hat{a}_1 = \frac{n(k+1)}{r_n^{k+1}}, \qquad \hat{a}_2 = \frac{m(k+1)}{s_m^{k+1}}, \tag{2.7}$$

respectively. Therefore using the invariance properties of the maximum likelihood estimation, the MLE of R becomes

$$\hat{R} = \frac{\hat{a}_2}{\hat{a}_1 + \hat{a}_2}.$$

To study the distribution of $\hat{R}$ we need the distributions of $\hat{a}_1$ and $\hat{a}_2$. Consider first $\hat{a}_1 = n(k+1)/r_n^{k+1}$, the pdf of the nth upper record value R_n is given by (Ahsanullah, 2004),

$$f_{R_n}(r_n) = \frac{1}{(n-1)!} f(r_n)[-\ln(1 - F(r_n))]^{n-1}. \tag{2.8}$$

Substituting f and F in (2.8), we obtain

$$f_{R_n}(r_n) = \frac{a_1^n}{(n-1)!(k+1)^{n-1}} r_n^{n(k+1)-1} \exp\left(-\frac{a_1}{k+1} r_n^{k+1}\right), \quad r_n > 0.$$

Consequently, the pdf of $Z_1 = \hat{a}_1$ is given by

$$f_{Z_1}(z_1) = \frac{(na_1)^n}{(n-1)! z_1^{n+1}} \exp\left(-\frac{na_1}{z_1}\right), \quad z_1 > 0. \tag{2.9}$$

This is recognized as the inverted gamma distribution, i.e., $Z_1 \sim IGamma(n, na_1)$. Similarly, the pdf of $Z_2 = \hat{a}_2$ is given by

$$f_{Z_2}(z_2) = \frac{(ma_2)^m}{(m-1)! z_2^{m+1}} \exp\left(-\frac{ma_2}{z_2}\right), \quad z_2 > 0. \tag{2.10}$$

Thus $Z_2 \sim IGamma(m, ma_2)$. Therefore we can find the pdf of

$$\hat{R} = \frac{\hat{a}_2}{\hat{a}_1 + \hat{a}_2} = \frac{Z_2}{Z_1 + Z_2} = \frac{1}{1 + \frac{Z_1}{Z_2}}.$$

Consider Z_1/Z_2. Note that, by the properties of the inverted gamma distribution and its relation with the gamma distribution we have $(na_1/Z_1) \sim Gamma(n,1)$ and $(ma_2/Z_2) \sim Gamma(m,1)$. Hence $(2na_1/Z_1) \sim \chi^2_{2n}$ and $(2ma_2/Z_2) \sim \chi^2_{2m}$. Note that, by the independence of two random quantities we have

$$\frac{(2ma_2/2mZ_2)}{(2na_1/2nZ_1)} = \frac{a_2 Z_1}{a_1 Z_2} \sim F_{(2m,2n)}.$$

Hence, $(Z_1/Z_2) = (a_1/a_2)F_{(2m,2n)}$, has a scaled F distribution. It follows that the distribution of $\hat{R}$ is that of $\frac{1}{1+(a_1/a_2)F_{(2m,2n)}}$ which can be obtained using simple transformation techniques. This fact can be used to construct the following $100(1-\alpha)\%$ confidence interval for R,

$$\left(\left(1+\frac{z_1}{z_2 F_{\alpha/2,2m,2n}}\right)^{-1}, \left(1+\frac{z_1}{z_2 F_{1-\alpha/2,2m,2n}}\right)^{-1}\right). \tag{2.11}$$

2.2 When the parameter k is unknown

If all of the parameters a_1, a_2 and k are unknown, in addition to the likelihood equations in (2.6), we must consider following likelihood equation

$$\begin{aligned}\frac{\partial \ell}{\partial k} &= -a_1 r_n^{k+1}\frac{(k+1)\ln r_n - 1}{(k+1)^2} - a_2 s_m^{k+1}\frac{(k+1)\ln s_m - 1}{(k+1)^2} \\ &\quad + \left(\sum_{i=1}^{n}\ln r_i + \sum_{i=1}^{m}\ln s_i\right) = 0.\end{aligned} \tag{2.12}$$

By replacing a_1 and a_2 obtained from (2.6) into (2.12) and after some simplification, we obtain the MLE of k as

$$\hat{k} = \left(\frac{n+m}{n\ln r_n + m\ln s_m - \sum_{i=1}^{n}\ln r_i - \sum_{i=1}^{m}\ln s_i}\right) - 1. \tag{2.13}$$

Consequently, the MLEs of a_1 and a_2 are given by

$$\hat{a}_1 = \frac{n(\hat{k}+1)}{r_n^{\hat{k}+1}}, \quad \hat{a}_2 = \frac{m(\hat{k}+1)}{s_m^{\hat{k}+1}}. \tag{2.14}$$

Hence, the MLE of R using (2.13) and (2.14) is given as

$$\hat{R} = \frac{\hat{a}_2}{\hat{a}_1 + \hat{a}_2}. \tag{2.15}$$

It is clear that the study of the distribution of $\hat{R}$ is very complicated and difficult to obtain. In this case, we will construct some confidence intervals based on the bootstrap method which are discussed in the next section.

3. Bootstrap Confidence Intervals

There are several bootstrap based intervals discussed in the literature (Efron and Tibshirani, 1993). In this section, we consider some parametric bootstrap confidence intervals for R. Firstly, we describe the procedure for generation of the bootstrap samples as follows:

Algorithm 3.1.

- *Step 1. Compute* $\hat{a}_1$, $\hat{a}_2$, $\hat{k}$ *and* $\hat{R}$, *the MLEs of* a_1, a_2, k *and* R *based on the original two samples of upper records* $\underset{\sim}{r}$ *and* $\underset{\sim}{s}$.
- *Step 2. Generate a bootstrap upper record sample* $\overset{*}{\underset{\sim}{r}} = (r_1^*, r_2^*, ..., r_n^*)$ *from PHRD*$(\hat{a}_1, \hat{k})$ *and similarly generate a bootstrap upper record sample* $\overset{*}{\underset{\sim}{s}} = (s_1^*, s_2^*, ..., s_m^*)$ *from PHRD*$(\hat{a}_2, \hat{k})$. *Based on these data, we compute the bootstrap estimates say,* $\hat{a}_1^*$, $\hat{a}_2^*$, $\hat{k}^*$ *and* $\hat{R}^*$.
- *Step 3. Repeat step 2, B times to obtain a set of bootstrap samples of R, say* $\hat{R}_1^*, ..., \hat{R}_B^*$.

Then we can compute the following bootstrap intervals:

Normal Interval: The simplest $100(1-\alpha)\%$ bootstrap interval is the Normal interval

$$(\hat{R} - z_{1-\alpha/2}\hat{s}e_{boot}, \hat{R} + z_{1-\alpha/2}\hat{s}e_{boot}) \tag{3.16}$$

where $\hat{s}e_{boot}$ is the bootstrap estimate of the standard error based on $\hat{R}_1^*, ..., \hat{R}_B^*$.

Percentile Interval: The $100(1-\alpha)\%$ bootstrap percentile interval is defined by

$$(\hat{R}^*_{(\alpha/2)B}, \hat{R}^*_{(1-\alpha/2)B}) \tag{3.17}$$

that is, just use the $\alpha/2$ and $1-\alpha/2$ quantiles of the bootstrap sample $\hat{R}_1^*, ..., \hat{R}_B^*$.

Studentized Pivotal (Student's t) Interval: Let

$$T_b^* = \frac{(\hat{R}_b^* - \hat{R})}{\hat{s}e_b^*}, \quad b = 1, 2, ..., B,$$

where $\hat{s}e_b^*$ is an estimate of the standard error of $\hat{R}_b^*$. Then the $100(1-\alpha)\%$ bootstrap Student's t interval is given by

$$(\hat{R} - t^*_{1-\alpha/2}\hat{s}e_{boot}, \hat{R} - t^*_{\alpha/2}\hat{s}e_{boot}) \tag{3.18}$$

where t_α^* is the α quantile of $T_1^*, ..., T_B^*$.

Interested readers may refer to DiCiccio and Efron (1996) and the references contained therein to observe more details.

4. Bayesian Inference

In this section, we discuss Bayesian estimation of R based on upper record values from the power hazard rate distribution in the following two cases:

4.1 When the parameter k is known

Under the assumption that the parameter k is known, the likelihood functions in (2.4) suggest a *Gamma* conjugate prior for a_1 and a_2 as

$$\pi(a_1) = \frac{\lambda_1^{\alpha_1} a_1^{\alpha_1 - 1} e^{-\lambda_1 a_1}}{\Gamma(\alpha_1)}, \quad a_1 > 0, \quad \alpha_1, \lambda_1 > 0,$$

$$\pi(a_2) = \frac{\lambda_2^{\alpha_2} a_2^{\alpha_2 - 1} e^{-\lambda_2 a_2}}{\Gamma(\alpha_2)}, \quad a_2 > 0, \quad \alpha_2, \lambda_2 > 0. \tag{4.19}$$

where α_1, λ_1, α_2 and λ_2 are the parameters of the prior distributions of a_1 and a_2, respectively. Combining these prior distributions with the likelihood functions in (2.4), the posterior distributions of a_1 and a_2 are given as

$$a_1 | \underset{\sim}{r} \sim Gamma(n + \alpha_1, \upsilon_1), \quad a_2 | \underset{\sim}{s} \sim Gamma(m + \alpha_2, \upsilon_2), \tag{4.20}$$

where

$$\upsilon_1 = \left(\lambda_1 + \frac{1}{k+1} r_n^{k+1} \right), \quad \upsilon_2 = \left(\lambda_2 + \frac{1}{k+1} s_m^{k+1} \right).$$

Since the priors a_1 and a_2 are independent, then, using standard transformation techniques and after some manipulations, the posterior pdf of R will be

$$f_R(r) = C \frac{r^{m+\alpha_2-1}(1-r)^{n+\alpha_1-1}}{[\upsilon_2 r + \upsilon_1(1-r)]^{n+m+\alpha_1+\alpha_2}}, \quad 0 < r < 1,$$

where

$$C = \frac{\Gamma(n+m+\alpha_1+\alpha_2)}{\Gamma(n+\alpha_1)\Gamma(m+\alpha_2)} \upsilon_1^{n+\alpha_1} \upsilon_2^{m+\alpha_2}.$$

The Bayes estimator under squared error loss is the mean of this posterior distribution, which can not be computed analytically. Alternatively, using the approximate method of Lindley (1980), it can be seen that the approximate Bayes estimator of R, say $\tilde{R}_B$, relative to squared error loss function is

$$\tilde{R}_B = \tilde{R} \left(1 + \frac{(1-\tilde{R})^2}{m + \alpha_2 - 1} - \frac{\tilde{R}(1-\tilde{R})}{n + \alpha_1 - 1} \right), \tag{4.21}$$

where $\tilde{R} = \frac{\tilde{a}_2}{\tilde{a}_1+\tilde{a}_2}$ and

$$\tilde{a}_1 = \frac{n+\alpha_1-1}{\upsilon_1}, \quad \tilde{a}_2 = \frac{m+\alpha_2-1}{\upsilon_2},$$

are the mode of the posterior densities a_1 and a_2, respectively. Furthermore, from the posterior densities a_1 and a_2, we obtain that $2\upsilon_1 a_1 | \underset{\sim}{r} \sim \chi^2_{2(n+\alpha_1)}$ and $2\upsilon_2 a_2 | \underset{\sim}{s} \sim \chi^2_{2(m+\alpha_2)}$. It follows that $\pi(R|\underset{\sim}{r}, \underset{\sim}{s})$, the posterior distribution of R, is equal to that of $(1 + AW)^{-1}$, where $W \sim F_{2(n+\alpha_1),2(m+\alpha_2)}$ and $A = \frac{\upsilon_2(n+\alpha_1)}{\upsilon_1(m+\alpha_2)}$. Therefore a Bayesian $100(1-\alpha)\%$ confidence interval for R is given by

$$\left((AF_{1-\alpha/2,2(n+\alpha_1),2(m+\alpha_2)} + 1)^{-1}, (AF_{\alpha/2,2(n+\alpha_1),2(m+\alpha_2)} + 1)^{-1}\right). \quad (4.22)$$

4.2 When the parameter k is unknown

In this subsection, we consider the Bayes estimation of R under assumption that all of the parameters (a_1, a_2, k) are unknown. It is assumed that a_1 and a_2 have conjugate priors $Gamma(\alpha_1, \lambda_1)$ and $Gamma(\alpha_2, \lambda_2)$ as mentioned in (4.19), respectively. It is also assumed that $(k + 1)$ has a prior $Gamma(\alpha_3, \lambda_3)$, i.e.

$$\pi(k) = \frac{\lambda_3^{\alpha_3}(k+1)^{\alpha_3-1}e^{-\lambda_3(k+1)}}{\Gamma(\alpha_3)}, \quad k > -1, \quad \alpha_3, \lambda_3 > 0. \quad (4.23)$$

Moreover, we assume that a_1, a_2 and k are independent. Therefore the joint posterior density of a_1, a_2 and k is given by

$$\pi(a_1, a_2, k|\underset{\sim}{r}, \underset{\sim}{s}) = \frac{L(a_1, a_2, k|\underset{\sim}{r}, \underset{\sim}{s})\pi(a_1)\pi(a_2)\pi(k)}{\int_0^\infty\int_0^\infty\int_{-1}^\infty L(a_1, a_2, k|\underset{\sim}{r}, \underset{\sim}{s})\pi(a_1)\pi(a_2)\pi(k)dkda_1da_2}, \quad (4.24)$$

where the numerator by using (2.4), (4.19) and (4.23) is given by

$$\frac{\lambda_1^{\alpha_1}a_1^{n+\alpha_1-1}}{\Gamma(\alpha_1)}e^{-\upsilon_1 a_1}\left(\prod_{i=1}^n r_i^k\right)\frac{\lambda_2^{\alpha_2}a_2^{m+\alpha_2-1}}{\Gamma(\alpha_2)}e^{-\upsilon_2 a_2}\left(\prod_{i=1}^m s_i^k\right)\frac{\lambda_3^{\alpha_3}(k+1)^{\alpha_3-1}}{\Gamma(\alpha_3)}e^{-\lambda_3(k+1)}.$$

Since the expression for $\pi(a_1, a_2, k|\underset{\sim}{r}, \underset{\sim}{s})$ in (4.24) can not be written in a closed form, we need a simulation technique to compute the Bayes estimate of R and the corresponding credible interval of R. We adopt the Gibbs sampling technique which use the posterior distributions of each parameter conditional on all others (see Gelfand and Smith, 1990). The conditional posterior distributions of a_1, a_2 and k can be obtained as follows:

$$(a_1|a_2, k, \underset{\sim}{r}, \underset{\sim}{s}) \sim Gamma(n + \alpha_1, \upsilon_1),$$

$$(a_2|a_1, k, \underset{\sim}{r}, \underset{\sim}{s}) \sim Gamma(m + \alpha_2, \upsilon_2)$$

and

$$\pi(k|a_1, a_2, \underset{\sim}{r}, \underset{\sim}{s}) \propto e^{-\upsilon_1 a_1} \left(\prod_{i=1}^{n} r_i^k \right) e^{-\upsilon_2 a_2} \left(\prod_{i=1}^{m} s_i^k \right) (k+1)^{\alpha_3 - 1} e^{-\lambda_3 (k+1)}. \quad (4.25)$$

Based on these conditional posterior distributions, we can easily generate samples of a_1 and a_2 from gamma densities. However, the conditional posterior distribution of k can not be reduced analytically to a well known distribution and therefore it is not possible to sample directly by standard methods. To do this, we use the Metropolis-Hastings method (Metropolis et al., 1953, and Hastings, 1970) with normal proposal distribution. Therefore, the algorithm of Gibbs sampling is described as follows:

Algorithm 4.1.

- *Step 1. Start with* $k^{(0)} = \hat{k}$ *as an initial guess and set* $t = 1$.
- *Step 2. Generate* $a_1^{(t)}$ *from* $Gamma(n + \alpha_1, \upsilon_1)$.
- *Step 3. Generate* $a_2^{(t)}$ *from* $Gamma(m + \alpha_2, \upsilon_2)$.
- *Step 4. Using Metropolis-Hastings method, generate* $k^{(t)}$ *from* $\pi(k|a_1, a_2, \underset{\sim}{r}, \underset{\sim}{s})$ *with the* $N(k^{(t-1)}, 1)$ *proposal distribution.*
- *Step 5. Compute* $R^{(t)} = a_2^{(t)}/(a_1^{(t)} + a_2^{(t)})$.
- *Step 6. Set* $t = t + 1$.
- *Step 7. Repeat Steps 2–6,* N *times.*

Now the approximate posterior mean and posterior variance of R become

$$\hat{E}(R|\underset{\sim}{r}, \underset{\sim}{s}) = \frac{1}{N-M} \sum_{t=M+1}^{N} R^{(t)} \quad (4.26)$$

and

$$\hat{V}(R|\underset{\sim}{r}, \underset{\sim}{s}) = \frac{1}{N-M} \sum_{t=M+1}^{N} \left(R^{(t)} - \hat{E}(R|\underset{\sim}{r}, \underset{\sim}{s})\right)^2, \quad (4.27)$$

where M is the burn-in period (that is, a number of iterations before the stationary distribution is achieved).

Based on N and $R^{(t)}$ values, using the method proposed by Chen and Shao (1999), a $100(1-\gamma)\%$ HPD credible interval can be constructed as

$$\left(R_{\left[\frac{\gamma}{2}N\right]}, R_{\left[\left(1-\frac{\gamma}{2}\right)N\right]}\right), \tag{4.28}$$

where $R_{\left[\frac{\gamma}{2}N\right]}$ and $R_{\left[\left(1-\frac{\gamma}{2}\right)N\right]}$ are the $\left[\frac{\gamma}{2}N\right]$-th smallest integer and the $\left[\left(1-\frac{\gamma}{2}\right)N\right]$-th smallest integer of $\{R^{(t)}, t = M+1, M+2, ..., N\}$, respectively .

5. An Illustrative Example

In order to illustrate the inferences discussed in the previous sections, in this section, we simulate 7 upper record values from PHRD$(2,1)$ and 7 upper record values from PHRD$(4,1)$. Therefore, $R_{Exact} = 0.6667$. The data has been truncated after four decimal places and it has been presented below. The $\underset{\sim}{r}$ upper record values are

$$1.0806, 1.5817, 1.6027, 2.5134, 2.5271, 2.5335, 2.5354$$

and the corresponding $\underset{\sim}{s}$ upper record values are

$$0.7028, 0.7924, 0.9578, 1.4263, 1.4430, 1.6930, 1.7296.$$

Case I: when k is known, we obtain the MLEs of a_1 and a_2 from (2.7) as, 2.1778 and 4.6799, respectively. Therefore, the MLE of R becomes $\hat{R} = 0.6824$. The corresponding 95% confidence interval based on (2.11) is equal to (0.4190,0.8449). Letting $\alpha_1 = \alpha_2 = 2$ and $\lambda_1 = \lambda_2 = 4$ in (4.21), we obtain $\tilde{a}_1 = 1.1089$, $\tilde{a}_2 = 1.4556$ and $\tilde{R} = 0.5675$. Therefore, the approximate Bayes estimator of R becomes $\tilde{R}_B = 0.5632$. The corresponding Bayesian 95% confidence interval based on (4.22) is equal to (0.3359,0.7731).

Case II: when k is unknown, we obtain the MLEs of k, a_1 and a_2 from (2.13) and (2.14) as, 2.1364, 1.1865 and 3.9376, respectively. Therefore, the MLE of R becomes $\hat{R} = 0.7684$. Based on 1000 bootstrap samples, the 95% bootstrap confidence intervals from (3.16), (3.17) and (3.18) are obtained as, (0.5458,0.9909), (0.5345,0.9681) and (0.5686,1), respectively. In Bayesian computation, we used hyper parameters $\alpha_1 = \alpha_2 = \alpha_3 = 2$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The approximate Bayes estimator of R from (4.26) based on $N = 5000$ samples and discard the first 1000 values as burn-in period becomes 0.6874. Also, the 95% HPD credible interval from (4.28) is obtained as, (0.4616,0.8648). The simulated values of R and Histogram of R generated by the algorithm of Gibbs sampling are plotted in Figure 1.

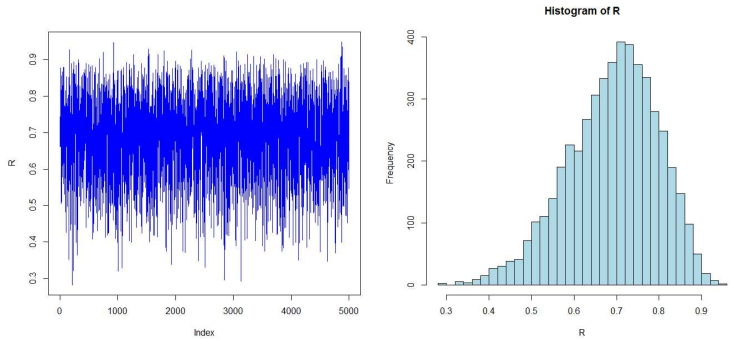

Figure 1: Simulated values of R and Histogram of R

6. A Simulation Study

Since the performance of the different methods proposed in the previous sections can not be compared theoretically, in this section, we present some results based on Monte Carlo simulations to compare the performance of the different methods. We compare the MLEs and Bayes estimators in terms of their biases and mean squared errors (MSEs). We also compare different confidence intervals in terms of their coverage probability and expected length. We consider two cases, namely when (I) the parameter k is known and (II) the parameter k is unknown, separately. In both cases we use all combinations of $n = 4, 5, 6$ and $m = 4, 5, 6$. We use the parameter values $a_1 = 1, 3$, $a_2 = 1$ and $k = 2$. Therefore, $R_{Exact} = 0.25, 0.5$. For computing the Bayes estimators and HPD credible intervals, we assume two priors as follows:

Prior 1: $\alpha_1 = \alpha_2 = \alpha_3 = \lambda_1 = \lambda_2 = \lambda_3 = 0$,

Prior 2: $\alpha_1 = \alpha_2 = \alpha_3 = 2$ and $\lambda_1 = \lambda_2 = \lambda_3 = 3$.

Note that Prior 1 is non-informative prior, while Prior 2 is an informative prior. We report all the results based on 1000 replications.

Case I: in this case, we obtain the average biases and MSEs of the MLE and the approximate Bayes estimator of R based on two Priors 1 and 2. We also compute the coverage probability and expected length for the confidence intervals obtained by using the ML and Bayesian methods. The results are reported in Tables 1 and 2.

Case II: in this case, first, we obtain the MLE of k and then compute the average biases and MSEs of the MLE and the approximate Bayes estimator of

Table 1: Biases and MSEs (in parentheses) of the estimators of R (k known)

(n, m)	R_{Exact}	MLE	Bayes (Prior 1)	Bayes (Prior 2)
(4,4)	0.25	0.0144(0.0135)	0.0395(0.0133)	0.1502(0.0278)
(4,5)		0.0099(0.0132)	0.0353(0.0130)	0.1548(0.0257)
(4,6)		0.0044(0.0124)	0.0308(0.0123)	0.1601(0.0243)
(5,4)		0.0143(0.0131)	0.0326(0.0128)	0.1270(0.0230)
(5,5)		0.0145(0.0123)	0.0336(0.0121)	0.1353(0.0220)
(5,6)		0.0116(0.0109)	0.0317(0.0110)	0.1388(0.0195)
(6,4)		0.0144(0.0134)	0.0377(0.0133)	0.1143(0.0196)
(6,5)		0.0185(0.0117)	0.0330(0.0112)	0.1198(0.0182)
(6,6)		0.0162(0.0103)	0.0315(0.0108)	0.1244(0.0165)
(4,4)	0.5	0.0007(0.0219)	0.0006(0.0173)	-0.0006(0.0067)
(4,5)		-0.0017(0.0211)	0.0017(0.0163)	0.0084(0.0062)
(4,6)		-0.0104(0.0203)	-0.0030(0.0159)	0.0112(0.0061)
(5,4)		0.0137(0.0216)	0.0088(0.0171)	-0.0027(0.0069)
(5,5)		0.0020(0.0196)	0.0019(0.0160)	0.0012(0.0061)
(5,6)		0.0045(0.0183)	0.0068(0.0151)	0.0095(0.0059)
(6,4)		0.0151(0.0198)	0.0074(0.0158)	-0.0094(0.0063)
(6,5)		0.0022(0.0188)	-0.0008(0.0152)	-0.0059(0.0058)
(6,6)		-0.0066(0.0156)	-0.0060(0.0131)	-0.0045(0.0056)

R based on two Priors 1 and 2. We also compute the coverage probability and expected length for the bootstrap confidence intervals, namely the normal interval, the percentile interval (Boot-p), the Student's t interval (Boot-t), and the HPD credible interval. For computing the bootstrap confidence intervals, we use 500 bootstrap iterations. We also compute the Bayes estimates and HPD credible intervals based on $N = 2000$ samples and discard the first 200 values as the burn-in period. The simulation results are reported in Tables 3 and 4.

7. Conclusions

In this article, we considered the problem of estimating the stress-strength reliability based on upper record values from the power hazard rate distribution. We used maximum likelihood approach and Bayesian approach for the estimation of R in two cases (I) when the parameter k is known and (II) when the parameter k is unknown.

Based on simulation results, we observe that the MSE and the expected length of the estimators decreas as sample sizes n and m increasing. From Tables 1 and 3,

Table 2: Expected lengths and coverage rates (in parentheses) of the confidence intervals with (1-α)=0.95 (k known)

(n,m)	R_{Exact}	MLE	Bayes (Prior 1)	Bayes (Prior 2)
(4,4)	0.25	0.5064(0.979)	0.5064(0.979)	0.5072(0.986)
(4,5)		0.4835(0.964)	0.4835(0.964)	0.4928(0.944)
(4,6)		0.4683(0.964)	0.4683(0.964)	0.4826(0.917)
(5,4)		0.4723(0.962)	0.4723(0.962)	0.4796(0.986)
(5,5)		0.4555(0.960)	0.4555(0.960)	0.4668(0.963)
(5,6)		0.4416(0.970)	0.4416(0.970)	0.4556(0.934)
(6,4)		0.4588(0.968)	0.4588(0.968)	0.4597(0.992)
(6,5)		0.4346(0.968)	0.4346(0.968)	0.4455(0.968)
(6,6)		0.4181(0.959)	0.4181(0.959)	0.4340(0.942)
(4,4)	0.5	0.5966(0.979)	0.5966(0.979)	0.5226(1.000)
(4,5)		0.5682(0.959)	0.5682(0.959)	0.5038(0.999)
(4,6)		0.5526(0.962)	0.5526(0.962)	0.4908(0.995)
(5,4)		0.5690(0.957)	0.5690(0.957)	0.5043(0.998)
(5,5)		0.5432(0.963)	0.5432(0.963)	0.4863(1.000)
(5,6)		0.5248(0.968)	0.5248(0.968)	0.4719(0.997)
(6,4)		0.5538(0.970)	0.5538(0.970)	0.4915(0.999)
(6,5)		0.5240(0.954)	0.5240(0.954)	0.4719(0.999)
(6,6)		0.5072(0.969)	0.5072(0.969)	0.4583(1.000)

we observe that the bias and MSE of the estimators are very small. These tables show that the performance of the MLE and the approximate Bayes estimator based on Prior 1 is almost the same but the performance of the approximate Bayes estimator based on Prior 2 is different. It appears that the MLE and the approximate Bayes estimator based on Prior 1 have the better performance for small values of R while the approximate Bayes estimator based on Prior 2 performs very well for values of R close to 0.5. When the parameter k is known, we observe from Table 2 that the performance of all confidence intervals is almost similar in terms of expected length, but in terms of coverage rate the Bayes interval based on Prior 2 has a better performance especially for values of R close to 0.5. When the parameter k is unknown, Table 4 shows that the performance of the bootstrap confidence intervals is very different than the Bayes intervals. We observe that the bootstrap confidence intervals have very short expected lengths in comparison with the Bayes intervals, but they have a low coverage rate. It appears that the performance of the Bayes interval based on Prior 2 is quite satisfactory, especially for values of R close to 0.5.

Table 3: Biases and MSEs (in parentheses) of the estimators of R (k unknown)

(n,m)	R_{Exact}	MLE	Bayes (Prior 1)	Bayes (Prior 2)
(4,4)	0.25	-0.0513(0.0302)	-0.0296(0.0265)	0.1834(0.0355)
(4,5)		-0.0639(0.0255)	-0.0373(0.0222)	0.1959(0.0325)
(4,6)		-0.0704(0.0252)	-0.0420(0.0218)	0.2001(0.0308)
(5,4)		-0.0424(0.0241)	-0.0277(0.0216)	0.1478(0.0280)
(5,5)		-0.0512(0.0213)	-0.0319(0.0188)	0.1589(0.0277)
(5,6)		-0.0376(0.0200)	-0.0178(0.0182)	0.1731(0.0271)
(6,4)		-0.0194(0.0226)	-0.0107(0.0205)	0.1267(0.0214)
(6,5)		-0.0291(0.0197)	-0.0161(0.0180)	0.1360(0.0208)
(6,6)		-0.0300(0.0191)	-0.0154(0.0176)	0.1422(0.0195)
(4,4)	0.5	-0.0076(0.0439)	-0.0066(0.0380)	-0.0018(0.0045)
(4,5)		-0.0307(0.0419)	-0.0236(0.0365)	0.0160(0.0039)
(4,6)		-0.0483(0.0362)	-0.0374(0.0317)	0.0266(0.0033)
(5,4)		0.0402(0.0429)	0.0332(0.0380)	-0.0130(0.0038)
(5,5)		-0.0072(0.0337)	-0.0068(0.0321)	-0.0032(0.0034)
(5,6)		-0.0184(0.0340)	-0.0142(0.0305)	0.0116(0.0032)
(6,4)		0.0407(0.0390)	0.0299(0.0336)	-0.0292(0.0045)
(6,5)		0.0282(0.0312)	0.0231(0.0298)	-0.0086(0.0039)
(6,6)		0.0093(0.0288)	0.0086(0.0262)	0.0031(0.0035)

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Table 4: Expected lengths and coverage rates (in parentheses) of the confidence intervals with (1-α)=0.95 (k unknown)

(n,m)	R_{Exact}	Normal	Boot-p	Boot-t	Bayes (Prior1)	Bayes (Prior2)
(4,4)	0.25	0.2932(0.678)	0.2629(0.656)	0.2543(0.618)	0.4395(0.834)	0.5284(0.976)
(4,5)		0.2800(0.726)	0.2491(0.728)	0.2467(0.689)	0.4371(0.832)	0.5195(0.910)
(4,6)		0.2701(0.775)	0.2484(0.783)	0.2458(0.724)	0.4274(0.818)	0.5116(0.834)
(5,4)		0.2835(0.738)	0.2574(0.741)	0.2528(0.692)	0.4178(0.830)	0.4986(0.998)
(5,5)		0.2716(0.800)	0.2536(0.795)	0.2459(0.710)	0.4133(0.836)	0.4905(0.964)
(5,6)		0.2652(0.843)	0.2476(0.826)	0.2407(0.804)	0.4134(0.832)	0.4842(0.868)
(6,4)		0.2805(0.780)	0.2542(0.776)	0.2518(0.714)	0.4049(0.854)	0.4740(0.998)
(6,5)		0.2693(0.836)	0.2493(0.823)	0.2474(0.815)	0.4019(0.844)	0.4652(0.980)
(6,6)		0.2608(0.878)	0.2428(0.854)	0.2403(0.851)	0.3955(0.848)	0.4579(0.930)
(4,4)	0.5	0.2883(0.715)	0.2624(0.694)	0.2617(0.805)	0.5708(0.882)	0.5290(1.000)
(4,5)		0.2774(0.723)	0.2537(0.703)	0.2504(0.809)	0.5493(0.886)	0.5121(1.000)
(4,6)		0.2592(0.729)	0.2391(0.701)	0.2376(0.796)	0.5457(0.900)	0.4983(1.000)
(5,4)		0.2761(0.722)	0.2519(0.710)	0.2494(0.814)	0.5457(0.854)	0.5122(1.000)
(5,5)		0.2428(0.730)	0.2271(0.722)	0.2274(0.830)	0.5288(0.886)	0.4934(1.000)
(5,6)		0.2124(0.754)	0.2163(0.736)	0.2072(0.841)	0.5123(0.892)	0.4784(1.000)
(6,4)		0.2428(0.733)	0.2350(0.709)	0.2332(0.809)	0.5425(0.888)	0.4985(1.000)
(6,5)		0.2015(0.767)	0.2019(0.730)	0.2005(0.835)	0.5173(0.894)	0.4796(1.000)
(6,6)		0.1944(0.781)	0.1733(0.748)	0.1713(0.856)	0.4940(0.904)	0.4637(1.000)

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