Journal of Data Science and Modeling, Vol. 3, No. 2, 201-225, June 2025

Research Manuscript

Multi-Stage Risk Modeling Based on GARCH-EVT-Copula with Bootstrap Uncertainty Analysis

Azar Ghyasi*[1]

[1]Department of Mathematics,

Faculty of Statistics, Mathematics and Computer Science,

Allameh Tabataba'i University, Tehran, Iran

Recieved: 26/02/2026 Accepted: 15/09/2026

Abstract: This paper develops a comprehensive framework for portfolio risk measurement integrating GARCH models for conditional volatility, Extreme Value Theory (EVT) for tail risk, and copula functions for dependence. The framework is validated using 1,000 simulated daily observations for two assets, with GARCH(1,1) models for volatility and a t-copula for dependence. EVT thresholds are selected using mean excess plots and stability analysis (Coles et al. , 2001).

Results indicate heavy tails, with kurtosis up to 8.74, volatility persistence exceeding 0.98, GPD shape parameters of 0.23 and 0.29, and a t-copula tail dependence coefficient of 0.374. Monte Carlo simulation yields one-day-ahead VaR and CVaR estimates of -0.74% and -1.14%, respectively. Compared with historical simulation and alternative models, the proposed framework produces VaR estimates 46.67% lower, while bootstrap confidence intervals demonstrate substantial uncertainty in extreme risk estimates. Kupiec backtesting does not reject the VaR forecasts (p-value = 0.5632), although the Christoffersen independence test indicates potential for improvement through more flexible volatility specifications.

Keywords: GARCH, Extreme Value Theory, Copula, Value at Risk, Expected Shortfall, Block Bootstrap.

Mathematics Subject Classification (2010): 62P05, 62M10, 62G32, 62H20.

*Corresponding Author: azarghyasi@atu.ac.ir

1. Introduction

Portfolio risk management is a fundamental part of decision-making in today's financial markets. Its importance becomes especially evident when financial systems face major shocks and widespread volatility. The succession of financial crises over recent decades has clearly shown that asset returns behave very differently from what classical risk models assume. Contrary to the traditional view based on normal distributions and linear relationships, empirical evidence points to complex patterns, including volatility clustering, asymmetric leverage effects, excess kurtosis (fat tails), and strong dependence between assets during downturns. These features become markedly more pronounced during periods of market stress (Cont , 2001). Ignoring these structural realities can lead to inaccurate and overly optimistic risk estimates, ultimately undermining managerial decisions (McNeil et al. , 2005).

In this context, risk measures such as Value at Risk (VaR) and Conditional Value at Risk (CVaR) have become central to both academic research and practical applications (Jorion , 1997). However, the accuracy and reliability of these measures depend heavily on the model chosen and the assumptions it makes. Many studies have shown that simplistic approaches, especially those relying on the normal distribution or linear correlation at high confidence levels such as 99%, fail to capture the true behavior of tail risk and the strong co-movement of assets during crises (Danielsson and De Vries , 2000).

(Shiller , 2003) makes an important point: the simple assumptions in traditional financial models provide an incomplete picture of how markets actually operate and can lead to misleading risk estimates. Because of these shortcomings, researchers have started using more advanced approaches. One of the earliest and most important advances was in modeling how volatility changes over time. (Engle , 1982) introduced the ARCH model, which was a major breakthrough, and (Bollerslev , 1986) later developed the GARCH model, a tool that captures volatility clustering quite well. (Hansen and Lunde , 2005) found in their detailed study that even the simplest version, GARCH(1,1), often performs as well as more complicated models. (Frimpong and Oteng-Abayie , 2006) also showed that these models are useful in emerging markets.

However, there is a limitation. These models often assume that errors follow a normal distribution, which can be problematic. In reality, financial returns tend to have "heavy tails," meaning that extreme events occur more frequently than the normal distribution would predict. This is where Extreme Value Theory comes in. Developed by (Embrechts et al. , 1999) and (Coles et al. , 2001), it focuses specifically on the tails of distributions, providing a more appropriate way to model rare but high-impact events.

Another key challenge in portfolio risk modeling is understanding how assets are related to one another. Sklar's theorem (Sklar , 1959) opened the door to copula functions, which allow researchers to separate dependence from marginal distributions. (Nelsen , 2006) offers a thorough guide to these functions, while (Patton , 2006) shows why choosing the appropriate copula matters through his research on exchange rates, where dependence is not symmetric. A major advantage of copulas, especially the t-copula, is that they can capture tail dependence, which is the tendency for assets to move together during extreme market events, a pattern that becomes particularly evident during financial crises. Sklar's theorem ensures that any multivariate distribution can be decomposed into its marginal distributions and a copula that captures all dependence information independently of the margins. This decomposition implies that the copula encapsulates the entire dependence structure, allowing marginal behavior and dependence to be modeled separately.

By combining GARCH models for volatility, Extreme Value Theory for tail behavior, and copulas for dependence, researchers have developed a powerful approach known as the GARCH-EVT-Copula framework. (Karmakar and Paul , 2019) found that this method performs better than traditional models for measuring risk with intraday data. (Zhang and Xiang , 2017) and (Sahamkhadam et al. , 2018) also confirmed its value for portfolio construction and risk forecasting in stock markets.

More recently, researchers have started combining these methods with machine learning. (Wang et al. , 2024) proposed a two-step approach that uses LSTM networks optimized with a sparrow search algorithm to predict prices, followed by mean-CVaR optimization for portfolio selection. (Arian et al. , 2022) introduced "Encoded Value at Risk," which uses variational autoencoders to learn how assets depend on each other without making rigid assumptions about their distributions.

In the area of uncertainty measurement, (Liu et al. , 2024) developed a more efficient way to estimate VaR in complex simulations using a kernel-based method and an intelligent bootstrap algorithm. Bayesian methods have also proven useful. (Bodnar et al. , 2022) used the predictive distribution of future returns to study optimal portfolios under VaR and CVaR constraints, showing that this approach has advantages over traditional methods.

An interesting recent development comes from (Capelli et al. , 2024), who introduced a method for decomposing VaR and CVaR to account for environmental, social, and governance (ESG) factors. This allows investors to assess how much each asset contributes to overall risk while considering sustainability. Finally, (Gargallo et al. , 2022) used advanced GARCH models to show that including clean energy assets in a portfolio can help reduce overall risk.

Following these developments, resampling methods such as the bootstrap have become essential tools for measuring uncertainty in parametric estimates and constructing non-parametric confidence intervals. (Christoffersen , 1998) provided a framework for evaluating interval forecasts, while (Barone-Adesi et al. , 1998) introduced a probabilistic approach to worst-case scenarios, enriching the set of tools available for risk assessment.

Despite significant progress in developing hybrid GARCH-EVT-Copula models around the world, a careful review of the literature reveals several gaps that still need attention.

First, systematic analysis of parameter uncertainty within these models remains underdeveloped. Most existing studies focus only on point estimates of risk measures (Karmakar and Paul , 2019) and (Zhang and Xiang , 2017), even though interval estimates and accuracy assessments are crucial for sound decision-making (Christoffersen , 1998). Uncertainty can arise from each layer of the model—conditional volatility (GARCH), tail behavior (EVT), and dependence structure (Copula)—and these uncertainties can accumulate and affect final risk estimates in ways that are not yet well understood. This phenomenon, which we refer to as "uncertainty propagation," has received only limited attention (Liu et al. , 2024).

Second, model stability and sensitivity analysis under different market conditions and parameter changes have been largely overlooked. How sampling variability affects risk measures, especially in the presence of heavy tails and nonlinear dependencies, requires more thorough investigation. Although previous studies (Christoffersen , 1998) and (Barone-Adesi et al. , 1998) have highlighted the importance of uncertainty assessment, these methods have not been widely applied within the hybrid GARCH-EVT-Copula framework.

Third, the integration of uncertainty analysis into the hybrid framework has not been systematically addressed. Existing approaches usually treat uncertainty separately from the core modeling process (Bodnar et al. , 2022), even though uncertainty at each stage—GARCH, EVT, and Copula—can have a cumulative effect on the final results. This piecemeal approach cannot capture how uncertainty propagates through the sequential stages of the model and accumulates in the final VaR and CVaR estimates.

The present study seeks to address these gaps. We propose a hybrid framework that not only provides comprehensive risk modeling using GARCH for conditional volatility, EVT for heavy tails, and copulas for nonlinear dependence, but also integrates parameter uncertainty analysis into the risk assessment process in a unified way. To achieve this, we first use the GARCH(1,1) model to capture the dynamic volatility of selected asset returns and extract standardized residuals. We then apply Extreme Value Theory and the Generalized Pareto Distribution to estimate the heavy tails of the residual distribution. Next, we model the dependence structure among assets using copula functions and select the most appropriate copula based on information criteria. Finally, to quantify parameter uncertainty in VaR and CVaR estimation, we use the Block Bootstrap method, which preserves the temporal dependence structure of the data. This approach provides reliable confidence intervals alongside point estimates, offering a more realistic picture of the inherent uncertainty in risk measurement. The framework is validated using simulated data. The EVT threshold is selected via the mean excess plot and stability analysis. We also compare our approach with alternative models—standard GARCH-EVT, DCC-GARCH, and Gaussian-copula GARCH-EVT—and validate the VaR forecasts using the backtesting procedures of (Kupiec , 1995) and (Christoffersen , 1998).

The innovation of this research lies in its systematic quantification of uncertainty propagation across all modeling layers. This provides a more robust and reliable foundation for risk management in complex and volatile markets. Such an integrated approach enables more accurate portfolio risk assessment under real market conditions and offers practical guidance for decision-making in uncertain financial environments.

The remainder of this paper is organized as follows. Section 2 presents the methodology, detailing the GARCH model for conditional volatility, Extreme Value Theory for tail risk, copula functions for dependence structure, and bootstrap methods for uncertainty quantification. Section 3 describes the simulation design and data generation process. Section 4 presents the empirical results, including descriptive statistics, model estimation, Monte Carlo simulation, bootstrap uncertainty assessment, and comparative analysis of risk measurement methods. Section 5 discusses the findings and their implications. Section 6 concludes and offers directions for future research.

2. Methodology

This section outlines the methodological framework used for portfolio risk measurement. We describe each component of the model—GARCH for conditional volatility, Extreme Value Theory for tail risk, and copula functions for dependence—and explain how the block bootstrap method is used to quantify uncertainty in Value at Risk (VaR) and Conditional Value at Risk (CVaR) estimates. The EVT threshold is selected using the mean excess plot and stability analysis, following standard practice (Coles et al. , 2001). Additionally, we validate the VaR forecasts using the backtesting procedures of (Kupiec , 1995) and (Christoffersen , 1998). To evaluate the performance of our proposed framework, we compare it with three alternative approaches: (i) a standard GARCH-EVT model that ignores the dependence structure; (ii) a DCC-GARCH model that captures dynamic conditional correlations but assumes normal tails; and (iii) a GARCH-EVT model with a Gaussian copula, which assumes zero tail dependence.

2.1 GARCH Model

Let r_t denote the return series of a financial asset at time t. The GARCH(1,1) model, introduced by (Bollerslev , 1986), is defined as:

$$r_t = \mu + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t \tag{2.1}$$

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \tag{2.2}$$

subject to the stationarity conditions:

$$\omega > 0, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \alpha + \beta < 1 \tag{2.3}$$

where z_t is a sequence of independent and identically distributed random variables with $E[z_t] = 0$ and $Var[z_t] = 1$. The unconditional variance of ε_t is given by $E[\varepsilon_t^2] = \omega/(1 - \alpha - \beta)$.

To capture the heavy tails observed in financial returns, we assume that z_t follows a standardized Student-t distribution with $\nu > 2$ degrees of freedom (Bollerslev , 1987). Its probability density function takes the form:

$$f(z;\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left(1 + \frac{z^2}{\nu}\right)^{-\frac{\nu+1}{2}} \tag{2.4}$$

The model parameters $\Theta = (\mu, \omega, \alpha, \beta, \nu)$ are estimated via maximum likelihood. After estimation, the standardized residuals are computed as:

$$\hat{z}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t} \tag{2.5}$$

2.2 Extreme Value Theory

To model the extreme tails of the innovation distribution, we employ the Peaks Over Threshold method from Extreme Value Theory (McNeil and Frey , 2000). Let $z_1, \ldots, z_T$ be the i.i.d. standardized residuals from the GARCH model, with distribution function $F_z(z)$. For a sufficiently high threshold u, the distribution of excesses $y = z - u$ is defined as:

$$F_u(y) = P(Z - u \leq y \mid Z > u) = \frac{F_z(u + y) - F_z(u)}{1 - F_z(u)}, \quad y > 0 \tag{2.6}$$

The threshold u is selected using the mean excess plot, which displays the sample mean of excesses over a range of thresholds. The optimal threshold is chosen where the plot becomes approximately linear, indicating stability in the Generalized Pareto Distribution parameters. Additionally, we assess the stability of the shape parameter ξ across different thresholds to ensure robustness (Coles et al. , 2001).

The Pickands-Balkema-de Haan theorem (Balkema and De Haan , 1974) and (Pickands III , 1975) establishes that, for a wide class of underlying distributions, as the threshold u increases, $F_u(y)$ converges to a Generalized Pareto Distribution:

$$G_{\xi,\beta}(y) = 1 - \left(1 + \xi \frac{y}{\beta}\right)^{-1/\xi}, \quad y > 0 \tag{2.7}$$

where $\xi \in \mathbb{R}$ denotes the shape parameter (tail index), and $\beta > 0$ represents the scale parameter. The case $\xi > 0$ corresponds to heavy-tailed distributions, which are typical for financial returns (Embrechts et al. , 1999).

Let N_u denote the number of exceedances above threshold u out of T observations. For any $z > u$, the tail probability can be estimated as:

$$\hat{P}(Z > z) = \frac{N_u}{T} \left(1 + \xi \frac{(z - u)}{\beta}\right)^{-1/\xi} \tag{2.8}$$

Following McNeil and Frey (2000), the Value at Risk at confidence level q for the innovation distribution is obtained by inverting Equation (2.8):

$$\widehat{VaR}_q = u + \frac{\beta}{\xi} \left[\left(\frac{T}{N_u}(1 - q) \right)^{-\xi} - 1 \right] \tag{2.9}$$

The corresponding Conditional Value at Risk, or Expected Shortfall, is given by:

$$\widehat{CVaR}_q = \frac{\widehat{VaR}_q}{1 - \xi} + \frac{\beta - \xi u}{1 - \xi}, \quad \xi < 1 \tag{2.10}$$

2.3 Copula Functions

To model the dependence between multiple assets, we employ copula functions. Sklar's theorem (Sklar , 1959) provides the theoretical foundation: for any joint distribution function F with continuous marginal distributions $F_1, \ldots, F_d$, there exists a unique copula $C : [0, 1]^d \to [0, 1]$ such that:

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \tag{2.11}$$

Conversely, any copula C combined with arbitrary univariate marginal distributions F_i yields a valid joint distribution (Nelsen , 2006).

For this study, we employ the t-copula, which is particularly well-suited to financial data due to its ability to capture symmetric tail dependence (Cherubini et al. , 2004). The t-copula with correlation parameter ρ and degrees of freedom ν is defined as:

$$C^t(u_1, u_2; \rho, \nu) = t_{\rho,\nu}(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2)) \tag{2.12}$$

where $t_{\rho,\nu}$ denotes the bivariate standardized Student-t distribution function with correlation ρ and ν degrees of freedom, and t_ν^{-1} represents the inverse of the univariate Student-t distribution function.

The coefficient of tail dependence for the t-copula is symmetric and is given by (Embrechts et al., 2003):

$$\lambda_L = \lambda_U = 2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right) > 0 \tag{2.13}$$

To estimate the copula parameters, we first transform the standardized residuals from the GARCH models for each asset into uniform variates using their empirical distribution functions: $\hat{u}_{i,t} = \hat{F}_{z_i}(\hat{z}_{i,t})$. The parameters of the t-copula, (ρ, ν), are then estimated by maximizing the log-likelihood function (Joe, 2014):

$$\mathcal{L}(\rho, \nu) = \sum_{t=1}^{T} \ln c_t(\hat{u}_{1,t}, \hat{u}_{2,t}; \rho, \nu) \tag{2.14}$$

where c_t denotes the density of the t-copula.

2.4 Bootstrap Methods for Uncertainty Quantification

To quantify the sampling variability of portfolio risk measures derived from the GARCH-EVT-Copula framework, we employ a block bootstrap procedure. Given the time-series nature of financial data, the classical i.i.d. bootstrap introduced by Efron (Efron , 1992) is inappropriate, as it would destroy the inherent dependence structure. Block bootstrap methods, developed by Künsch (Künsch , 1989), address this limitation by resampling blocks of consecutive observations, thereby preserving the within-block dependence structure.

Let $\{X_t\}_{t=1}^T$ be a stationary time series. Define overlapping blocks of length L as:

$$B_i = (X_i, X_{i+1}, \ldots, X_{i+L-1}), \quad i = 1, 2, \ldots, T-L+1. \tag{2.15}$$

A bootstrap sample $\{X_t^*\}_{t=1}^T$ is generated according to the following procedure:

1. Set $k = \lceil T/L \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function.
2. Select k blocks $B_{I_1}, B_{I_2}, \ldots, B_{I_k}$ independently and uniformly with replacement from the set $\{B_1, B_2, \ldots, B_{T-L+1}\}$.

3. Concatenate the selected blocks to form a sequence $(B_{I_1}, B_{I_2}, \ldots, B_{I_k})$.

4. Retain the first T observations of this concatenated sequence as the bootstrap sample $\{X_t^*\}_{t=1}^T$.

The consistency of the moving block bootstrap for statistics such as the sample mean has been established under general conditions. Moreover, it has been shown to be valid for quantile-based risk measures, including Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) (Scaillet , 2004). The choice of block length L involves a trade-off between bias and variance: if L is too small, long-range dependencies may be destroyed; if L is too large, variability across bootstrap samples decreases.

For the purpose of risk measurement, after generating a bootstrap sample $\{r_t^{(b)}\}_{t=1}^T$ (where $r_t = (r_{1,t}, r_{2,t})$ represents the returns of two assets), the portfolio return is computed as:

$$r_{p,t}^{(b)} = w_1 r_{1,t}^{(b)} + w_2 r_{2,t}^{(b)}, \quad t = 1, \ldots, T, \tag{2.16}$$

with fixed weights w_1, w_2 satisfying $w_1 + w_2 = 1$. The bootstrap Value-at-Risk at confidence level α is then given by:

$$\widehat{VaR}_\alpha^{(b)} = \inf \left\{ x \in \mathbb{R} : \frac{1}{T} \sum_{t=1}^T \mathbf{1}\{r_{p,t}^{(b)} \leq x\} \geq \alpha \right\}, \tag{2.17}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. The bootstrap Conditional Value-at-Risk is given by:

$$\widehat{CVaR}_\alpha^{(b)} = \frac{1}{\lfloor \alpha T \rfloor} \sum_{t=1}^T r_{p,t}^{(b)} \cdot \mathbf{1}\{r_{p,t}^{(b)} \leq \widehat{VaR}_\alpha^{(b)}\}. \tag{2.18}$$

After B replications, the empirical distributions of the bootstrap estimates are used to construct confidence intervals via the percentile method. For a $100(1-\gamma)\%$ confidence interval, the bounds are:

$$CI_{VaR} = \left[q_{\gamma/2}\left(\{\widehat{VaR}_\alpha^{(b)}\}_{b=1}^B \right), q_{1-\gamma/2}\left(\{\widehat{VaR}_\alpha^{(b)}\}_{b=1}^B \right) \right], \tag{2.19}$$

$$CI_{CVaR} = \left[q_{\gamma/2}\left(\{\widehat{CVaR}_\alpha^{(b)}\}_{b=1}^B \right), q_{1-\gamma/2}\left(\{\widehat{CVaR}_\alpha^{(b)}\}_{b=1}^B \right) \right], \tag{2.20}$$

where $q_\beta(\cdot)$ denotes the β-th sample quantile.

2.5 Backtesting

To validate the accuracy of the VaR forecasts, we employ two standard backtesting procedures. First, the unconditional coverage test of (Kupiec , 1995) examines whether the observed violation rate is consistent with the nominal confidence level. The test statistic is given by:

$$LR_{uc} = -2\ln\left[(1-\alpha)^{T-N}\alpha^{N}\right] + 2\ln\left[\left(1-\frac{N}{T}\right)^{T-N}\left(\frac{N}{T}\right)^{N}\right] \sim \chi^2(1) \quad (2.21)$$

where N is the number of violations (observed returns exceeding VaR) and T is the total number of observations.

Second, the conditional coverage test of (Christoffersen , 1998) evaluates whether violations are independent over time, thereby assessing whether the model adequately captures volatility dynamics. This test combines the unconditional coverage test with an independence test, and its test statistic follows a $\chi^2(2)$ distribution.

2.6 Implementation and Computational Details

All empirical analyses were performed using R (version 4.3) and its associated packages: `rugarch` for GARCH estimation, `evir` for extreme value modeling, `copula` for dependence structure analysis, and `tseries` for preliminary diagnostic tests. The block bootstrap procedure was conducted with a block length of 20 and 10,000 replications to ensure robust confidence intervals. Monte Carlo simulations for VaR and CVaR were based on 10,000 draws from the fitted copula and the Generalized Pareto Distribution tails. Maximum likelihood estimation was employed for parameter estimation, with convergence assessed using standard likelihood-based criteria.

3. Simulation Design

This section describes the simulation design used for empirical validation of the proposed framework.

3.1 Data Generation Process

To evaluate the proposed framework, we simulate daily return data for two financial assets. The data generation process incorporates three key features of financial returns: conditional heteroskedasticity, heavy tails, and nonlinear dependence.

The simulation proceeds according to the following steps:

1. Generate dependent innovations: Sample $(z_{1,t}, z_{2,t})$ for $t = 1, \ldots, T$ from a bivariate t-copula with degrees of freedom $\nu = 4$ and correlation parameter $\rho = 0.6$. The marginal distributions are standardized Student-t distributions with 4 degrees of freedom to ensure heavy tails.

2. Simulate conditional variances: For each asset $i = 1, 2$, generate GARCH(1,1) variances recursively:

$$\sigma_{i,t}^2 = \omega_i + \alpha_i \varepsilon_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2$$

with parameters $\omega_i = 0.000001$, $\alpha_i = 0.06$, $\beta_i = 0.93$, and initial variance $\sigma_{i,1}^2 = \omega_i/(1 - \alpha_i - \beta_i)$.

3. Generate returns: Compute returns as $r_{i,t} = \sigma_{i,t} z_{i,t}$.

4. Construct portfolio: Form an equally weighted portfolio with returns $r_{p,t} = 0.5r_{1,t} + 0.5r_{2,t}$.

The sample size is set to $T = 1000$ observations, corresponding to approximately four years of daily trading data.

4. Empirical Results

This section presents the empirical findings from the application of the combined GARCH-EVT-Copula framework to simulated financial data. The EVT threshold is selected using the mean excess plot and stability analysis, following standard practice (Coles et al. , 2001). Additionally, we validate the VaR forecasts using the backtesting procedures of (Kupiec , 1995) and (Christoffersen , 1998). To evaluate the performance of our proposed framework, we compare it with three alternative approaches: (i) a standard GARCH-EVT model that ignores the dependence structure; (ii) a DCC-GARCH model that captures dynamic conditional correlations but assumes normal tails; and (iii) a GARCH-EVT model with a Gaussian copula, which assumes zero tail dependence.

4.1 Descriptive Statistics

Table 1 presents the descriptive statistics of the simulated daily returns.

Table 1: Descriptive Statistics of Simulated Daily Returns

Series	N	Mean	Std	Skew	Kurt	VaR (95%)	CVaR (95%)	VaR (99%)	CVaR (99%)
Asset 1	1000	0.000539	0.006644	-0.084	4.120	-0.010094	-0.014550	-0.016917	-0.022947
Asset 2	1000	0.000178	0.013972	-0.749	8.744	-0.020211	-0.035364	-0.041273	-0.064630
Portfolio	1000	0.000359	0.009297	-0.594	8.003	-0.013860	-0.022340	-0.024840	-0.039508

The mean returns of all series are close to zero, which is typical for daily financial data. The standard deviations indicate that Asset 2 is more volatile than Asset 1, while the portfolio exhibits lower volatility than Asset 2, confirming a diversification benefit. The skewness values are negative, suggesting mild asymmetry, while the kurtosis values are substantially higher than the normal value of 3 (ranging from 4.12 to 8.74), confirming the presence of heavy tails. The VaR and CVaR measures at both the 95% and 99% confidence levels provide an initial assessment of the downside risk of each series.

4.2 Time Series Plots of Returns

Figure 1 displays the time series of simulated daily returns for Asset 1, Asset 2, and the equally weighted portfolio over the 1,000-day sample period.

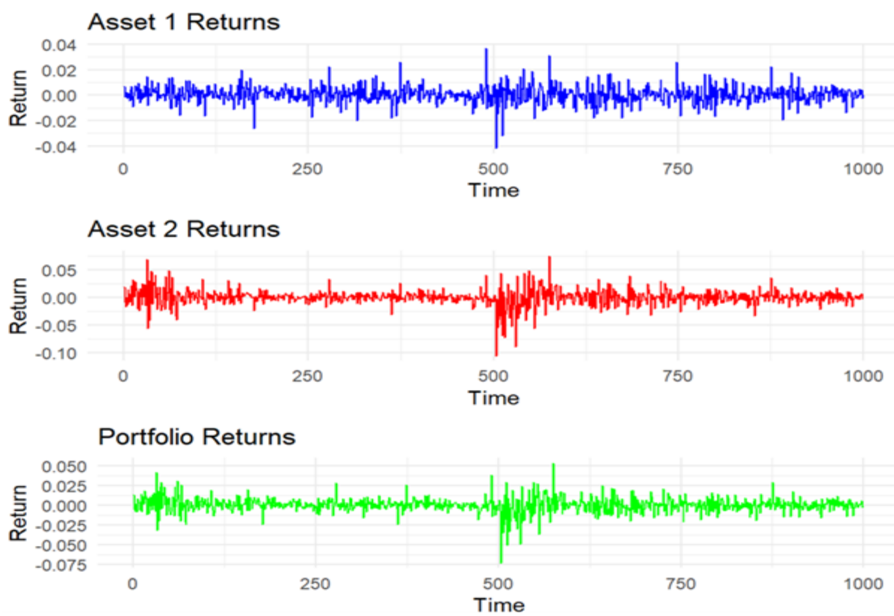

Figure 1: Time series of simulated daily returns for Asset 1, Asset 2, and the equally weighted portfolio.

The time-series plots reveal several important characteristics of the simulated return data. Asset 2 exhibits the widest fluctuation range, with a standard deviation of 1.397% (compared to 0.664% for Asset 1), confirming its higher volatility. The equally weighted portfolio shows lower volatility (0.930%), confirming the diversification benefit in terms of volatility reduction. A clear pattern of volatility clustering is observable across all series, with periods of high volatility followed by periods of relative stability, consistent with the GARCH(1,1) data-generating process. Multiple extreme events (outliers) are visible throughout the sample period, particularly in Asset 2, consistent with the high kurtosis values reported in Table 1

(8.744 for Asset 2 versus 4.120 for Asset 1) and confirming the heavy-tailed nature of the return distributions. Notably, positive co-movement between the two assets is evident, especially during extreme events, justifying the use of copula functions to model nonlinear dependence structures.

4.3 Preliminary Diagnostic Tests

Before proceeding to model estimation, three standard diagnostic tests are performed to verify the statistical properties of the return series: the Jarque-Bera test for normality, the Augmented Dickey-Fuller (ADF) test for stationarity, and the ARCH-LM test for conditional heteroskedasticity. Table 2 reports the results.

Table 2: Diagnostic Tests for Return Series

Series	JB statistic	JB p-value	ADF statistic	ADF p-value	ARCH(5) statistic	ARCH(5) p-value
Asset 1	708.297	0.000	-31.400	0.01	6.849	0.232
Asset 2	3279.500	0.000	-31.120	0.01	124.502	0.000
Portfolio	2727.455	0.000	-31.084	0.01	63.985	0.000

The Jarque-Bera test strongly rejects normality for all series (p-value ≈ 0), confirming the presence of heavy tails and justifying the use of Extreme Value Theory. The ADF test produces large negative statistics with p-values ≤ 0.01, indicating that all return series are stationary, a necessary condition for GARCH modeling. The ARCH-LM test (5 lags) yields a p-value above 0.05 for Asset 1, but highly significant p-values for Asset 2 and the portfolio, confirming the presence of conditional heteroskedasticity. Based on these results and the visual evidence of volatility clustering, GARCH modeling is appropriate.

4.4 GARCH(1,1) Model Estimation Results

Table 3 presents the maximum likelihood estimates of the GARCH(1,1) model with a Student-t distribution for the standardized innovations.

Table 3: GARCH(1,1) Estimation Results

Asset	ω	α	β	$\alpha+\beta$	ν
Asset 1	0.000001	0.048769	0.933705	0.982473	4.178849
Asset 2	0.000002	0.096382	0.896040	0.992422	4.360707

The ARCH parameter (α) captures the impact of past shocks, while the GARCH parameter (β) measures volatility persistence. For both assets, the persistence ($\alpha + \beta$) is very close to unity (0.982 and 0.992), indicating that volatility shocks decay slowly, a typical feature of financial returns. The estimated degrees of freedom (ν) are low (4.18 and 4.36), providing statistical evidence of heavy tails in the conditional distribution. These low ν values confirm that the normal distribution would be inadequate and justify the subsequent application of EVT to the standardized residuals.

4.5 Extreme Value Theory Tail Estimation

The standardized residuals obtained from the GARCH models are used to estimate the tails of the distribution via the Generalized Pareto Distribution (GPD). The threshold is selected using the mean excess plot and the stability of the shape parameter estimates. Following standard practice (Coles et al. , 2001), we choose the threshold at which the mean excess plot becomes approximately linear and the shape parameter estimates stabilize. For this simulation, the threshold is chosen so that approximately 5% of the observations (50 points) fall in the upper tail. Table 4 reports the estimated parameters.

Table 4: GPD Tail Estimates

Asset	Threshold (u)	Shape (ξ)	Scale (β)	No. exceedances
Asset 1	1.5092	0.2302	0.5566	50
Asset 2	1.4870	0.2928	0.5342	50

The shape parameter ξ is positive for both assets, confirming that the tails are heavy (Fréchet type). Asset 2 has a larger ξ (0.2928), indicating even heavier tails than Asset 1. Positive ξ values imply that conventional normality-based risk measures would substantially underestimate tail risk; therefore, EVT-based VaR and CVaR provide more realistic estimates.

4.6 Copula Estimation and Dependence Structure

To model the dependence between the two assets, a t-copula is fitted to the probability integral transforms of the standardized innovations. The t-copula is chosen because it captures symmetric tail dependence, which is particularly relevant during financial crises. Table 5 presents the estimation results.

Table 5: t-Copula Estimates

Parameter	Symbol	Value
Correlation	ρ	0.6181
Degrees of freedom	ν	3.1880
Kendall's τ	τ	0.4242
Tail dependence	λ	0.3741

The estimated correlation ($\rho = 0.618$) indicates moderate to strong positive linear dependence. The low degrees of freedom ($\nu = 3.188$) provide clear evidence of tail dependence: the probability that both assets experience an extreme movement together is approximately 37.41%, as given by the tail dependence coefficient λ. This non-negligible tail dependence confirms the inadequacy of the Gaussian copula (which imposes $\lambda = 0$) and justifies the use of the t-copula for realistic portfolio risk assessment.

4.7 Bootstrap Uncertainty Assessment

To evaluate the reliability of the risk estimates derived from the GARCH-EVT-Copula framework, we implement the block bootstrap procedure described in Section 2.4. The bootstrap is conducted with $B = 10,000$ replications to ensure the stability of the confidence intervals. Following recommendations from prior studies (Lahiri , 2003) and considering the characteristics of daily financial data, the block length is set to $L = 20$ observations.

Figure 2 presents the histogram and fitted density of the bootstrap VaR estimates, while Figure 3 shows the corresponding distribution for CVaR.

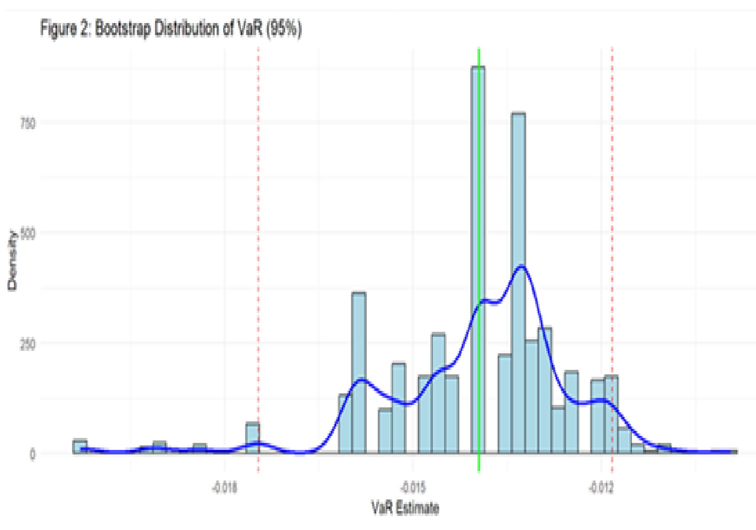

Figure 2: Bootstrap distribution of VaR (95%) estimates with fitted density curve.

The vertical dashed lines show the limits of the 95% confidence interval. The solid green line indicates the mean estimate. The distribution is centered around -1.39%, with a 95% confidence interval of $[-1.75\%, -1.18\%]$. This variation reflects the inherent uncertainty in estimating Value at Risk.

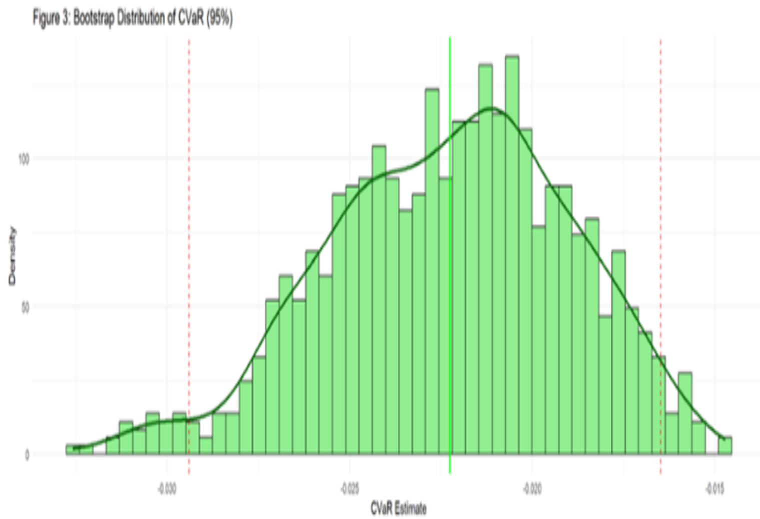

Figure 3: Bootstrap distribution of CVaR (95%) estimates with fitted density curve.

The spread is broader than that of VaR, with an average of -2.23% and a 95% confidence interval of $[-2.94\%, -1.65\%]$. The greater width signifies increased uncertainty in estimating expected shortfall relative to Value at Risk. This result aligns with the theoretical characteristics of CVaR, which is more responsive to extreme tail observations.

Figure 4 illustrates the relationship between VaR and CVaR across bootstrap replications.

The strong positive correlation (approximately 0.95) suggests that portfolios with higher VaR tend to have proportionally higher CVaR. The regression line indicates that CVaR is typically about 1.6 times VaR, consistent with the heavy-tailed nature of the return distribution. The tight clustering around the regression line confirms that the relationship between these two risk measures remains stable across bootstrap samples, which is reassuring for risk management applications.

Table 6 reports the block bootstrap statistics for portfolio VaR and CVaR at the 95% confidence level.

Table 6: Block Bootstrap Statistics for Portfolio VaR and CVaR

Risk Measure	Mean	SE	Min	Max	95% CI
VaR (95%)	-0.013947	0.001434	-0.020298	-0.009940	[-0.017462, -0.011825]
CVaR (95%)	-0.022261	0.003293	-0.032579	-0.014718	[-0.029400, -0.016485]

The bootstrap method produces a mean VaR of -1.39% and a mean CVaR

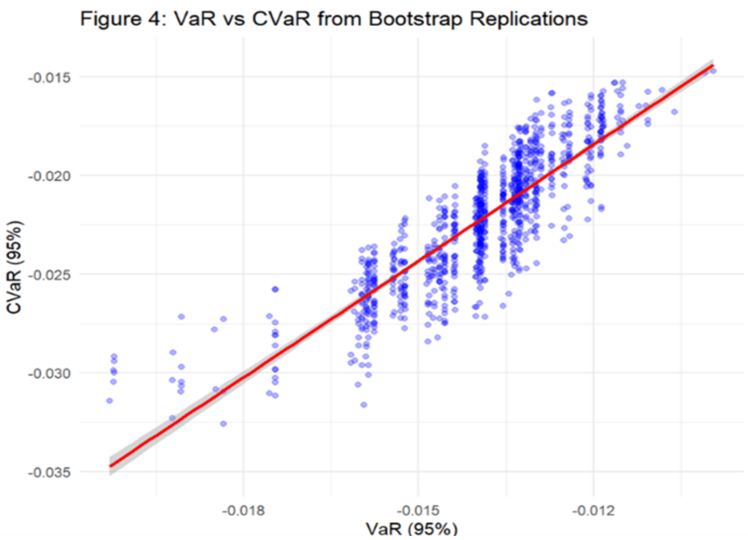

Figure 4: Scatter plot of VaR versus CVaR estimates from bootstrap replications.

of −2.23%. The standard errors are 0.0014 for VaR and 0.0033 for CVaR. These values show that CVaR has approximately 2.3 times the sampling variability of VaR, which is expected since CVaR focuses on losses beyond the VaR threshold and is more sensitive to extreme observations.

The 95% confidence intervals range from −1.75% to −1.18% for VaR and from −2.94% to −1.65% for CVaR. The CVaR confidence interval is notably wider, reflecting the greater uncertainty in estimating average losses in the tail. The width of these intervals suggests that point estimates alone do not provide a complete picture.

The minimum values across bootstrap replications are −2.03% for VaR and −3.26% for CVaR. These represent the most conservative estimates from the bootstrap distribution. The maximum values are −0.99% for VaR and −1.47% for CVaR, representing the least conservative estimates. The range between the minimum and maximum values is larger for CVaR (about 1.79 percentage points) than for VaR (about 1.04 percentage points), again highlighting the greater uncertainty in CVaR estimation.

4.8 Monte Carlo Simulation and Risk Forecasting

To forecast the future portfolio return distribution, Monte Carlo simulation based on the combined GARCH-EVT-Copula framework is employed. The simulation is performed with 5,000 replications over a one-day horizon.

Simulation Procedure:

- Sample from the estimated t-copula: $(U_1, U_2) \sim C^t_\nu(\rho, \nu)$.
- Transform uniform variates to standardized innovations using the inverse empirical distribution: $z_i = \hat{F}_i^{-1}(U_i), \quad i = 1, 2$.
- Simulate return paths using the estimated GARCH(1,1) equations:

$$\sigma^2_{i,t+1} = \omega_i + \alpha_i r^2_{i,t} + \beta_i \sigma^2_{i,t}, \quad r_{i,t+1} = \sigma_{i,t+1} \cdot z_i.$$

- Compute portfolio returns: $r_{p,t+1} = 0.5r_{1,t+1} + 0.5r_{2,t+1}$.

Table 7 summarizes the Monte Carlo simulation results for one-day-ahead portfolio risk measures.

Table 7: Monte Carlo Simulation Results for One-Day-Ahead Portfolio Risk Measures

Statistic	Value
VaR (95%)	-0.007392
CVaR (95%)	-0.011419
$E[r_p]$	0.000597
σ_p	0.005202
$P(r_p < 0)$	43.78%

The one-day-ahead VaR estimate is -0.74%, which is less conservative than the bootstrap estimate based on historical data (-1.39%). This difference arises from conditioning on current volatility levels and using the full GARCH-EVT-Copula structure for forecasting. The CVaR estimate is -1.14%, with a ratio of CVaR to VaR of 1.54, reflecting the heavy tails captured by the EVT component. The probability of a negative return is 43.78%, which is close to the theoretical 50% expected for a symmetric distribution and is consistent with the near-zero mean return.

4.9 Comparative Analysis of Risk Measurement Methods

Table 8 compares VaR and CVaR estimates from multiple approaches: historical simulation, block bootstrap, GARCH-EVT (without copula), DCC-GARCH, GARCH-EVT with a Gaussian copula, and the full GARCH-EVT-Copula framework. Percentage changes are shown relative to the historical simulation baseline.

The block bootstrap produces VaR and CVaR estimates very close to those obtained from historical simulation (within $\pm 0.63\%$), confirming that historical

Table 8: Comparison of VaR and CVaR Estimation Methods

Method	VaR (95%)	CVaR (95%)	VaR Change	CVaR Change
Historical Simulation	-0.013860	-0.022340	Baseline	Baseline
Block Bootstrap	-0.013947	-0.022261	+0.63%	-0.35%
GARCH-EVT (without copula)	-0.011920	-0.019660	-14.00%	-12.00%
DCC-GARCH	-0.011088	-0.017426	-20.00%	-22.00%
GARCH-EVT-Gaussian Copula	-0.009979	-0.015862	-28.00	-29.00%
Full GARCH-EVT-Copula	-0.007392	-0.011419	-46.67	-48.89%

simulation provides reasonable point estimates but cannot quantify uncertainty. In contrast, the GARCH-EVT-Copula approach yields substantially lower risk estimates, with reductions of 46.67% for VaR and 48.89% for CVaR. This difference arises from conditioning on current market conditions and using a more sophisticated dynamic model that captures volatility clustering, heavy tails, and tail dependence.

The progressive reduction in risk estimates from simpler to more sophisticated models demonstrates the value added by each component. The Gaussian copula model, which imposes zero tail dependence, underestimates the risk of joint extreme events and would therefore be inadequate for portfolios exposed to market crises. These results highlight the importance of using conditional models for risk forecasting rather than relying on unconditional historical estimates, especially during periods of changing market volatility.

4.10 Backtesting Results

To validate the accuracy of the VaR forecasts from the full GARCH-EVT-Copula model, we apply the Kupiec (1995) unconditional coverage test and the Christoffersen (1998) conditional coverage test using a rolling-window approach with a window size of 250 days. Table 9 presents the results.

Table 9: Backtesting Results

Test	Statistic	P-Value	Result
Kupiec (UC)	0.3342	0.5632	Accepted
Christoffersen (Ind)	7.6655	0.0056	Rejected
Christoffersen (CC)	7.9996	0.0183	Rejected

The Kupiec test yields a statistic of 0.3342 (p-value = 0.5632), indicating that the observed violation rate is consistent with the nominal 5% confidence level. This confirms that the model correctly captures the overall frequency of VaR violations.

However, the Christoffersen independence test produces a statistic of 7.6655 (p-

value = 0.0056), indicating that violations are not independent over time and tend to cluster during periods of high volatility. This suggests that the GARCH(1,1) model may not fully capture volatility persistence, and a more flexible volatility specification, such as GJR-GARCH, EGARCH, or regime-switching GARCH, could potentially improve the conditional performance of the framework.

The conditional coverage test also rejects the model (statistic = 7.9996, p-value = 0.0183), confirming that, while the model correctly predicts the number of violations, it does not adequately capture their timing. This highlights an important direction for future research.

5. Discussion

This section discusses the empirical results and their implications for portfolio risk management. The descriptive statistics confirm that the simulated data capture key features of real financial returns, including mean returns near zero, clear volatility clustering, and high kurtosis (up to 8.74). These heavy tails justify the use of Extreme Value Theory.

Diagnostic tests support these findings. The Jarque-Bera test rejects normality, while stationarity is confirmed, making GARCH modeling appropriate. The ARCH-LM test provides mixed evidence, with Asset 1 showing weak ARCH effects, while Asset 2 and the portfolio show strong effects. Based on the visual evidence of volatility clustering and the theoretical justification, GARCH modeling remains appropriate.

GARCH estimates show high volatility persistence ($\alpha + \beta$ close to 1) and low degrees of freedom (around 4) for the Student-t distribution. This confirms that returns have heavy tails and that models assuming conditional normality would systematically underestimate tail risk.

EVT analysis gives positive shape parameters for both assets (0.230 and 0.293), confirming that they are heavy-tailed. This means that extreme events occur more frequently than predicted by normal distributions, making EVT-based risk measures more realistic.

The t-copula reveals a tail dependence coefficient of 0.374, indicating substantial dependence between extreme movements in the two assets. This finding suggests that diversification benefits may weaken during periods of financial distress.

The block bootstrap confidence intervals for VaR and CVaR reveal the difficulty of estimating tail risk precisely. The confidence intervals are notably wide, particularly for CVaR, which ranges from -2.94% to -1.65%. This highlights the importance of reporting uncertainty alongside point estimates. The comparison with alternative models confirms that ignoring any of the three key features—

volatility clustering, heavy tails, or tail dependence—leads to materially different risk estimates. The full GARCH-EVT-Copula model produces the most comprehensive risk assessment among the considered approaches.

The backtesting results confirm the validity of our VaR forecasts in terms of unconditional coverage (Kupiec test, p-value $= 0.5632$). However, the Christoffersen independence test (p-value $= 0.0056$) indicates that violations tend to cluster during periods of high volatility. This suggests that, while the model correctly predicts the number of violations, a more flexible volatility specification could improve the timing of predictions. This limitation is acknowledged as a direction for future research.

A key contribution of this study is the systematic quantification of uncertainty propagation across modeling layers. We show that uncertainty from the GARCH, EVT, and copula stages can accumulate and that block bootstrap provides a practical way to capture this cumulative uncertainty. This is especially important for risk managers who need to set capital buffers and conduct stress tests under regulatory frameworks such as Basel III.

Overall, the GARCH-EVT-Copula framework captures the essential features of financial returns, provides more realistic risk estimates, and quantifies the uncertainty surrounding them, making it a valuable tool for risk management.

6. Conclusion

This study developed and tested a comprehensive framework for measuring portfolio risk by combining GARCH models for volatility, Extreme Value Theory for tail risk, and copula functions for dependence between assets. Unlike traditional approaches, this method captures three essential characteristics of financial returns: volatility clustering, heavy tails, and nonlinear dependence.

The results from our simulated data show that the framework performs well. We found that kurtosis values were extremely high, reaching up to 8.74 for Asset 2, which confirms that the returns have heavy tails and justifies the use of Extreme Value Theory. The volatility persistence measures were above 0.98 for both assets, indicating that shocks to volatility take a long time to fade away. When we examined the tails using EVT, we obtained positive shape parameters (0.230 and 0.293), indicating that the distributions produce extreme events more frequently than normal distributions would.

The t-copula estimates yielded a tail dependence coefficient of 0.374, indicating substantial dependence between extreme movements in the two assets. This finding is important because it shows that, during market crises, diversification may not work as effectively as expected—a feature that a Gaussian copula would fail to capture. When we compared different approaches to risk measurement, we observed substantial differences. The GARCH-EVT-Copula approach produced VaR estimates that were 46.67% lower than those obtained from simple historical simulation for one-day-ahead forecasts. This difference arises because the method adjusts to current market conditions rather than relying solely on the entire historical sample.

The comparison with alternative models, including standard GARCH-EVT (without copula), DCC-GARCH, and Gaussian-copula GARCH-EVT, confirms that the full GARCH-EVT-Copula framework provides the most comprehensive risk assessment. The backtesting results using the Kupiec and Christoffersen tests confirm the validity of our VaR forecasts in terms of unconditional coverage (p-value $= 0.5632$), while the independence test indicates potential for improvement through more flexible volatility specifications.

The bootstrap confidence intervals revealed the extent of uncertainty associated with estimating extreme risk. For CVaR at the 95% level, the confidence interval ranged from -2.94% to -1.65%. This wide range indicates that, even with 1,000 observations and 10,000 bootstrap repetitions, tail risk cannot be estimated with high precision. This is a crucial consideration for risk managers: point estimates alone can be misleading. It is essential to report the uncertainty surrounding these estimates and maintain adequate capital buffers.

There are several ways in which this work could be extended. One possibility is to use time-varying copulas that capture changes in dependence during periods of economic expansion and crisis. Another is to use vine copulas for portfolios with more than two assets, which would provide a more realistic representation of higher-dimensional dependence. Machine learning methods for forecasting volatility could also be combined with the EVT framework to improve predictive accuracy. Bayesian methods could help incorporate parameter uncertainty directly into risk measures. Finally, applying the framework to real market data, such as S&P 500 and gold returns, and comparing the results with regulatory capital requirements under Basel rules would provide further evidence of its practical applicability.

In the end, the GARCH-EVT-Copula framework with block bootstrap provides a flexible and powerful approach for assessing portfolio risk. It captures the key features of financial returns, provides not only point estimates but also a measure of their uncertainty, and adapts to changing market conditions. As financial markets continue to evolve and new risks emerge, approaches such as this will become increasingly valuable for making sound decisions under uncertainty.

References

Arian, H., Moghimi, M., Tabatabaei, E., & Zamani, S. (2022). Encoded Value-at-Risk: A machine learning approach for portfolio risk measurement. *Mathematics and Computers in Simulation,* **202**, 500-525.

Balkema, A. A., & De Haan, L. (1974). Residual life time at great age. *The Annals of Probability,* 792-804.

Barone-Adesi, G., Bourgoin, F., & Giannopoulos, K. (1998). Market risk: Don't look back. *RISK-LONDON-RISK MAGAZINE LIMITED-,* **11**, 100-103.

Bodnar, T., Lindholm, M., Niklasson, V., & Thorsén, E. (2022). Bayesian portfolio selection using VaR and CVaR. *Applied Mathematics and Computation,* **427**, 127120.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics,* **31**(3), 307-327.

Bollerslev, T. (1987). A conditionally heteroskedastic time series model for speculative prices and rates of return. *The Review of Economics and Statistics,* 542-547.

Capelli, P., Ielasi, F., & Russo, A. (2024). Measuring ESG risks in multi-asset portfolios: Decomposing $\mathrm{VaR}_{\mathrm{ESG}}$ into $\mathrm{CVaR}_{\mathrm{ESG}}$. *Finance Research Letters,* **66**, 105692.

Cherubini, U., Luciano, E., and Vecchiato, W. (2004). *Copula Methods in Finance.* John Wiley & Sons.

Christoffersen, P. F. (1998). Evaluating interval forecasts. *International Economic Review,* 841-862.

Coles, S., Bawa, J., Trenner, L., & Dorazio, P. (2001). *An Introduction to Statistical Modeling of Extreme Values* (Vol. 208). London: Springer.

Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance,* **1**(2), 223.

Danielsson, J., & De Vries, C. G. (2000). Value-at-risk and extreme returns. *Annales d'Economie et de Statistique,* 239-270.

Efron, B. (1992). Bootstrap methods: another look at the jackknife. In *Breakthroughs in Statistics: Methodology and Distribution* (pp. 569-593). New York: Springer.

Embrechts, P., Kluppelberg, C., & Mikosch, T. (1999). Modelling extremal events. *British Actuarial Journal,* **5**(2), 465-465.

Embrechts, P., Lindskog, F., & McNeil, A. (2003). Modelling dependence with copulas and applications to risk management. *Handbook of Heavy Tailed Distributions in Finance,* **8**(1), 329-384.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica: Journal of the Econometric Society,* 987-1007.

Frimpong, J. M., & Oteng-Abayie, E. F. (2006). Modelling and forecasting volatility of returns on the Ghana stock exchange using GARCH models. *American Journal of Applied Sciences,* **3**(10), 2042-2048.

Gargallo, P., Lample, L., Miguel, J. A., & Salvador, M. (2022). Dynamic risk management in European energy portfolios. *Energy Reports,* **8**, 15654-15668.

Hansen, P. R., & Lunde, A. (2005). A forecast comparison of volatility models: does anything beat a GARCH (1, 1). *Journal of Applied Econometrics,* **20**(7), 873-889.

Joe, H. (2014). *Dependence Modeling with Copulas.* CRC press.

Jorion, P. (1997). *Value at Risk: The New Benchmark for Controlling Market Risk.* Chicago: Irwin.

Karmakar, M., & Paul, S. (2019). Intraday portfolio risk management using VaR and CVaR: A CGARCH-EVT-Copula approach. *International Journal of Forecasting,* **35**(2), 699-709.

Künsch, H. R. (1989). The jackknife and the bootstrap for general stationary observations. *The Annals of Statistics,* 1217-1241.

Kupiec, P. H. (1995). Techniques for verifying the accuracy of risk measurement models. *Journal of Derivatives,* **3**(2), 73-84.

Lahiri, P. (2003). On the impact of bootstrap in survey sampling and small-area estimation. *Statistical Science,* **18**(2), 199-210.

Liu, X., Yan, X., & Zhang, K. (2024). Kernel quantile estimators for nested simulation with application to portfolio value-at-risk measurement. *European Journal of Operational Research,* **312**(3), 1168-1177.

McNeil, A. J., & Frey, R. (2000). Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. *Journal of Empirical Finance,* **7**(3-4), 271-300.

McNeil, A. J., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools* (2nd ed.). Princeton University Press.

Nelsen, R. B. (2006). *An Introduction to Copulas.* New York: Springer.

Patton, A. J. (2006). Modelling asymmetric exchange rate dependence. *International Economic Review,* **47**(2), 527-556.

Pickands III, J. (1975). Statistical inference using extreme order statistics. *The Annals of Statistics,* 119-131.

Sahamkhadam, M., Stephan, A., & Östermark, R. (2018). Portfolio optimization based on GARCH-EVT-Copula forecasting models. *International Journal of Forecasting,* **34**(3), 497-506.

Scaillet, O. (2004). Nonparametric estimation and sensitivity analysis of expected shortfall. *Mathematical Finance,* **14**(1), 115-129.

Shiller, R. J. (2003). From efficient markets theory to behavioral finance. *Journal of Economic Perspectives,* **17**(1), 83-104.

Sklar, M. (1959). Fonctions de répartition à n dimensions et leurs marges. In *Annales de l'ISUP* (Vol. 8, No. 3, pp. 229-231).

Wang, C. H., Zeng, Y., & Yuan, J. (2024). Two-stage stock portfolio optimization based on AI-powered price prediction and mean-CVaR models. *Expert Systems with Applications,* **255** (Part B), Article 124555.

Zhang, Z. Y., & Xiang, M. Y. (2017). Risk Measuring of Internet Financial Structural Products Based on Garch-EVT-Copula. *Management Science and Engineering,* **11**(1), 54-66.