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A goodness-of-fit test for progressively first-failure-censored data from a proportional hazard rate model

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Abstract: Progressive first-failure censoring schemes are potentially useful in practical applications where budget constraints exist or rapid testing is required. Moreover, several common sampling schemes such as first-failure censoring, progressive Type-II censoring, Type-II censoring, and complete sampling can be viewed as special cases of the progressive first-failure censoring scheme. In this article, we propose a goodness-of-fit test statistic to assess whether a progressively first-failure-censored sample originates from a distribution belonging to the proportional hazard rate model. This model encompasses several well-known lifetime distributions, including the exponential, Rayleigh, Lomax, Burr Type-XII, and Pareto distributions, among others. We derive the null distribution of the proposed test statistic. Through Monte Carlo simulations, we evaluate the power of the proposed test under a range of different alternative distributions, assuming the null distribution to be exponential, Rayleigh, or Pareto. Finally, we present several numerical real-world examples to demonstrate the practical applicability of the proposed goodness-of-fit test. We also summarize the main findings and provide concluding remarks.

Keywords: Goodness-of-fit test, Proportional hazard rate model, Progressive first-failure censoring, Monte Carlo simulation, Power study.

Mathematics Subject Classification (2010): 62N03, 62F03.

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1. Introduction

Parametric statistical procedures are appropriate under the assumption of a specific distribution for the random variable under study. Thus, it is an important issue to test the validity of the hypothesized distribution based on the sample data prior to any inferential procedure. The goodness-of-fit (GOF) technique has been discussed by numerous scholars, and the history of this technique began with the seminal article of Karl Pearson in 1900 on chi-squared tests. A comprehensive review of this topic can be found in D'Agostino and Stephens (1986).

Recently, the application of censored samples for making inferences about target populations has grown within reliability studies and life-testing experiments. This shift is largely due to modern technological advances, which have enabled the production of increasingly reliable products. When assessing the reliability of such products, waiting for every test unit to fail is often impractical for researchers. Instead, they can stop the experiment before all failures occur and collect data from only a subset of the units. The resulting sample is known as a censored sample. This method proves highly efficient in terms of cost, time, and labor.

The statistical literature outlines several censoring strategies, which are generally divided into single-stage and multi-stage censoring plans. Under single-stage censoring, the experiment stops once a fixed number of failures has been reached or a set time has elapsed, as seen in Type-I and Type-II censoring schemes. These plans do not allow for the removal of working units before the experiment ends. Multi-stage censoring plans, however, permit the withdrawal of functional units at any stage of the test, not just at its conclusion. A widely used example of a multi-stage approach is progressive Type-II censoring. For further details, see Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014).

The practical relevance of progressive censoring arises from several real-world scenarios:

1. **Wear and degradation studies**: In studies of wear, investigating the actual aging process requires items to be fully disassembled at various stages of the experiment. Progressive censoring accommodates this need by allowing intermediate removal of units, enabling researchers to examine physical degradation mechanisms without running the test to completion.

2. **Rapid screening and accelerated life-testing**: When the goal is to quickly assess product reliability, progressive censoring provides a compromise between reducing experimental duration and still observing a sufficient number of extreme failures. This makes it particularly useful in industrial settings where time-to-market considerations are critical.

3. **Conservation of expensive or scarce units**: In experiments involving costly or hard-to-obtain items, early removal of surviving units allows these items to be preserved and reused in subsequent tests. Progressive censoring facilitates this by permitting planned withdrawals before the experiment ends.

4. **Ethical considerations in biomedical and survival studies**: In clinical or survival experiments, ethical concerns may necessitate early removal of subjects from a study. Progressive censoring accommodates such withdrawals, which are not permitted under conventional Type-I or Type-II censoring schemes.

5. **Unavoidable loss of units**: In practical testing environments, units may be lost accidentally (e.g., breakage) or due to loss of contact with participants. Progressive censoring naturally handles such occurrences, as it allows for removals at any point during the experiment.

6. **Efficient use of testing facilities**: In industrial settings, freeing up testing equipment and resources before the end of a life-test can lead to significant cost and time savings. Progressive censoring enables intentional removal of units to optimize resource utilization.

Various extensions have been proposed to further generalize the progressive Type-II censoring scheme. For example, Wu and Kuş (2009) proposed a progressive first-failure censoring scheme, under which the progressive Type-II censoring plan can be derived as a special case. In this scheme, n disjoint groups, each containing k identical units (so that $N = n \times k$), are placed on a life test at time zero. The test is terminated at the time of the m-th failure. When the i-th failure occurs (for $i = 1, \ldots, m-1$), R_i randomly selected groups, along with the group containing the i-th failure, are removed from the test. Once the m-th failure is observed, all remaining groups are removed from the test.

Notice that m and $(R_1, \ldots, R_m)$ are fixed prior to the study. It needs to be mentioned that:

1. If $k = 1$, the progressive first-failure censoring scheme reduces to progressive Type-II censoring.

2. If $R_i = 0$ for $i = 1, \ldots, m$, this censoring scheme corresponds to the first-failure censoring scheme.

3. If $k = 1$, $R_i = 0$ for $i = 1, \ldots, m-1$ and $R_m = n - m$, the Type-II censoring scheme is obtained.

4. If $k = 1$, $R_i = 0$ for $i = 1, \ldots, m$, the progressively first-failure-censored data are reduced to a complete sample.

In recent years, GOF tests under progressive censoring have received the attention of many authors. For example, Michael and Schucany (1979) used a transformation method to transfer progressively Type-II censored data to a reduced set of uncensored data and then performed a standard GOF test for uniformity on the transformed data. Balakrishnan *et al.* (2002) proposed a GOF test for the exponential model when the available sample is progressively Type-II censored. Balakrishnan and Lin (2003) derived the exact null distribution of the test statistic proposed by Balakrishnan *et al.* (2002). Balakrishnan *et al.* (2004) extended their method to general location-scale families of distributions. Wang (2008) proposed another GOF test for the exponential distribution based on spacings from progressively Type-II censored data. Noughabi and Chahkandi (2018) and Noughabi and Noughabi (2025) proposed a general GOF test based on an estimate of the Kullback-Leibler information for hybrid Type-I censored data and progressively Type-II censored data, respectively, and the power of the proposed tests was assessed under the exponentiality assumption. Recently, Kong *et al.* (2025) proposed several GOF tests for progressively Type-II censored data based on the cumulative hazard function. These tests were exemplified using the two-parameter Rayleigh distribution. Interested readers may also refer to Balakrishnan *et al.* (2007), Habibi Rad *et al.* (2011), Pakyari and Balakrishnan (2013), Döring and Cramer (2019), Ahrari and Habibi Rad (2021), Bhati *et al.* (2025), and Batsidis *et al.* (2026) for other developments.

Throughout this article, we suppose that the distribution function (DF) of lifetimes F satisfies the proportional hazard rate (PHR) model. Let X and Y be two random variables with hazard rate functions h_F and h_G, respectively. Then X and Y are said to satisfy the PHR model, proposed by Cox (1972), with proportionality constant $\theta > 0$, if $h_F(x) = \theta\, h_G(x)$ for all x. Or, equivalently,

$$\bar{F}(x) = \left(\bar{G}(x)\right)^{\theta}, \tag{1.1}$$

for all x, where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$. This model includes several well-known lifetime distributions such as the exponential, Rayleigh, Lomax, Burr Type-XII, Pareto, Weibull, and Gompertz, among others. Also, this model is flexible enough to accommodate both monotonic and non-monotonic failure rates, even when the baseline failure rate is monotonic. For further details on PHR models, one may refer to Marshall and Olkin (2007).

In this article, we propose a GOF test for the PHR model when the available sample is progressively first-failure-censored. Hence, the rest of the article is organized as follows. In Section 2, we describe the proposed GOF test and derive the exact null distribution for the test statistic. The power of the proposed test is then assessed through Monte Carlo simulations in Section 3, when the null distribution is exponential, Rayleigh, or Pareto. Section 4 is dedicated to the applicability of the proposed GOF test on some real data sets.

2. Proposed test

Let $\boldsymbol{X} = (X_{1:m:n:k}, \ldots, X_{m:m:n:k})$ be the progressively first-failure-censored sample with $(R_1, \ldots, R_m)$ denoting the progressive censoring scheme. We would like to test the null hypothesis H_0: the sample $\boldsymbol{X}$ comes from a specified distribution belonging to the PHR model with survival function as given in (1.1).

Define the spacings

$$\begin{aligned} Z_1 &= nW_1, \\ Z_2 &= (n - R_1 - 1)(W_2 - W_1), \\ &\vdots \\ Z_m &= (n - R_1 - \cdots - R_{m-1} - m + 1)(W_m - W_{m-1}), \end{aligned} \tag{2.2}$$

where $W_i = -k \ln\left(\bar{G}(X_{i:m:n:k})\right)$ for $i = 1, \ldots, m$. It has been proved that the spacings $Z_1, \ldots, Z_m$, as defined in (2.2), are independent and identically distributed as exponential with parameter θ; see Theorem 2.7 of Balakrishnan and Aggarwala (2000).

Let us define the test statistic

$$T = -2 \sum_{i=1}^{m-1} \ln \frac{Z_1 + \cdots + Z_i}{Z_1 + \cdots + Z_m}. \tag{2.3}$$

It is clear that small and large values of T lead to the rejection of the null hypothesis.

Define

$$V_i = \frac{Z_1 + \cdots + Z_i}{Z_1 + \cdots + Z_m},$$

for $i = 1, \ldots, m-1$. Balakrishnan *et al.* (2002) proved that the joint distribution of $V_1, \ldots, V_{m-1}$ is the same as the joint distribution of the $m-1$ order statistics $U_{(1)}, \ldots, U_{(m-1)}$ generated from a random sample of size $m-1$ from the standard uniform distribution, denoted by $U_1, \ldots, U_{m-1}$. Therefore, we have

$$T = 2 \sum_{i=1}^{m-1} (-\ln V_i) \stackrel{d}{=} 2 \sum_{i=1}^{m-1} \left(-\ln U_{(i)}\right) \stackrel{d}{=} 2 \sum_{i=1}^{m-1} (-\ln U_i)\,.$$

Hence, the distribution of the test statistic T under the null hypothesis is a chi-squared distribution with $2m-2$ degrees of freedom. The power function of the proposed test is

$$P\left(T < \chi^2_{\alpha/2,2m-2}\middle|H_1\right) + P\left(T > \chi^2_{1-\alpha/2,2m-2}\middle|H_1\right),$$

where $\chi^2_{\alpha,m}$ denotes the α-th lower quantile of the chi-squared distribution with m degrees of freedom. The procedure of the proposed test is summarized in Algorithm 1 using the censored data $(X_{1:m:n:k}, \ldots, X_{m:m:n:k})$ and censoring scheme $(R_1, \ldots, R_m)$.

Algorithm 1 Computational procedure for the proposed test

1. Compute $W_i = -k \ln\left(\bar{G}(X_{i:m:n:k})\right)$ for $i = 1, \ldots, m$.
2. Calculate the spacings $Z_1, \ldots, Z_m$ using (2.2).
3. Based on the spacings $Z_1, \ldots, Z_m$, compute the test statistic T using (2.3).
4. Reject the null hypothesis at significance level α if $T < \chi^2_{\alpha/2,2m-2}$ or $T > \chi^2_{1-\alpha/2,2m-2}$.

In reliability and life-testing applications, the exponential, Rayleigh, and Pareto distributions are extensively used for modelling lifetime data; see, e.g., Lawless (2003). Therefore, in this article, we specifically consider the following three distributions. However, the test discussed here is applicable to any distribution in the PHR model.

1. Exponential distribution with DF $F(x) = 1 - \exp\{-\theta x\},\ x > 0$.

2. Rayleigh distribution with DF $F(x) = 1 - \exp\left\{-\theta x^2\right\},\ x > 0$.

3. Pareto distribution with DF $F(x) = 1 - x^{-\theta},\ x > 1$.

Also, note that the test statistic T in Equation (2.3) is clearly scale invariant under the Exponential and Rayleigh distributions. In order to be scale invariant for the Pareto distribution, the test statistic T may be modified as

$$T_\star = -2\sum_{i=2}^{m-1} \ln \frac{Z_2 + \cdots + Z_i}{Z_2 + \cdots + Z_m}.$$

It is obvious that the distribution of $T_\star$ under the null hypothesis is a chi-squared distribution with $2m-4$ degrees of freedom. Small and large values of $T_\star$ indicate that the null hypothesis should be rejected. Hence, the power function of $T_\star$ is given by

$$P\left(T_\star < \chi^2_{\alpha/2,2m-4}\middle|H_1\right) + P\left(T_\star > \chi^2_{1-\alpha/2,2m-4}\middle|H_1\right).$$

3. Simulation study

In this section, we assess the power of the proposed test when testing for the exponential, Rayleigh, and Pareto null distributions against different alternatives. For testing the exponential and Rayleigh null distributions, the following lifetime distributions are used as alternatives to the null distribution:

- Weibull distribution $\mathrm{W}(\theta)$ with density function

$$f(x) = \theta x^{\theta-1} \exp\left\{-x^{\theta}\right\}, \; x > 0, \; \theta > 0.$$

- Log-normal distribution $\mathrm{LN}(\theta)$ with density function

$$f(x) = \frac{1}{\theta x \sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log x}{\theta}\right)^2\right\}, \; x > 0, \; \theta > 0.$$

- Gamma distribution $\mathrm{G}(\theta)$ with density function

$$f(x) = \frac{x^{\theta-1} \exp\{-x\}}{\Gamma(\theta)}, \; x > 0, \; \theta > 0.$$

For testing the Pareto null distribution, the gamma, log-normal, log-gamma, log-Weibull, and Weibull distributions were used as alternatives. The density functions of the log-gamma $\mathrm{LG}(\theta)$ and log-Weibull $\mathrm{LW}(\theta)$ distributions are

$$f(x) = \frac{(\log x)^{\theta-1}}{x^2\, \Gamma(\theta)}, \; x > 1, \; \theta > 0,$$

and

$$f(x) = \frac{\theta\, (\log x)^{\theta-1}}{x} \exp\left\{-\left(\log x\right)^{\theta}\right\}, \; x > 1, \; \theta > 0,$$

respectively.

Table 1 (see Appendix) presents the different choices of progressive censoring schemes used in the simulation study. The level of significance is 0.1. We also compare the power performance of the proposed test against that of alternative tests. Pakyari and Balakrishnan (2012) proposed goodness-of-fit tests for progressively Type-II censored data based on the empirical distribution function. They specifically considered modified Kolmogorov-Smirnov, Cramér-von Mises, and Anderson-Darling tests. These tests can be used for any continuous null distribution $F_\theta(\cdot)$. On the basis of the progressively Type-II censored data $(X_{1:m:n}, \ldots, X_{m:m:n})$ and censoring scheme $(R_1, \ldots, R_m)$, the procedure for these tests is presented in Algorithm 2.

By setting $R_i^\star = k(R_i + 1) - 1$ for $i = 1, \ldots, m$, the test statistics D, W, and A can be adapted for progressively first-failure censored data. Then, in order to

Algorithm 2 Computational procedure for the modified Kolmogorov-Smirnov, Cramér-von Mises, and Anderson-Darling tests

1. Find the maximum likelihood estimate of the unknown parameter θ, denoted by $\hat{\theta}$, under the hypothesized distribution; and calculate $X^{\star}_{i:m:n} = F_{\hat{\theta}}(X_{i:m:n})$ for $i = 1, \ldots, m$.
2. Calculate $Y_{i:m:n} = \Phi^{-1}(X^{\star}_{i:m:n})$ for $i = 1, \ldots, m$, where $\Phi^{-1}(\cdot)$ denotes the quantile function of the standard normal distribution.
3. Considering $Y_{1:m:n}, \ldots, Y_{m:m:n}$ as progressively Type-II censored data from a normal distribution with mean μ and standard deviation σ, calculate the maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}$.
4. Calculate $Y^{\star}_{i:m:n} = \Phi\{(Y_{i:m:n} - \hat{\mu})/\hat{\sigma}\}$ for $i = 1, \ldots, m$.
5. Calculate D, W, and A, where these are the modified Kolmogorov-Smirnov, Cramér-von Mises, and Anderson-Darling test statistics, respectively; and according to Pakyari and Balakrishnan (2012) are as follows

$$D = \max\Big\{\max_{1\leq i\leq m}\{\alpha_{i:m:n} - Y^{\star}_{i:m:n}\}, \max_{1\leq i\leq m}\{Y^{\star}_{i:m:n} - \alpha_{i-1:m:n}\}\Big\}, \tag{3.4}$$

$$W = n\sum_{i=1}^{m-1}\alpha_{i:m:n}(Y^{\star}_{i+1:m:n} - Y^{\star}_{i:m:n})(\alpha_{i:m:n} - Y^{\star}_{i+1:m:n} - Y^{\star}_{i:m:n}) + \frac{n}{3}Y^{\star 3}_{m:m:n}, \tag{3.5}$$

$$A = n\sum_{i=1}^{m-1}\left\{\alpha^2_{i:m:n}\log\left(\frac{Y^{\star}_{i+1:m:n}(1 - Y^{\star}_{i:m:n})}{Y^{\star}_{i:m:n}(1 - Y^{\star}_{i+1:m:n})}\right) + 2\alpha_{i:m:n}\log\left(\frac{1 - Y^{\star}_{i+1:m:n}}{1 - Y^{\star}_{i:m:n}}\right)\right\} - n\big(\log(1 - Y^{\star}_{m:m:n}) + Y^{\star}_{m:m:n}\big), \tag{3.6}$$

where

$$\alpha_{i:m:n} = 1 - \prod_{k=m-i+1}^{m}\frac{k + \sum_{j=m-i+1}^{m}R_j}{1 + k + \sum_{j=m-i+1}^{m}R_j}, \quad \text{for} \quad i = 1, \ldots, m.$$

6. Reject the null hypothesis at the significance level α if the test statistic exceeds the upper tail significance points.

compute the power of these test statistics under progressive first-failure censoring, it is necessary to use the censoring scheme $(R^{\star}_1, \ldots, R^{\star}_m)$ instead of $(R_1, \ldots, R_m)$ in Equations (3.4), (3.5), and (3.6). For each censoring scheme in Table 1, we compare the power values of the proposed test with those of the competing tests. All power estimates are based on 10,000 simulation replications, and the corresponding standard error is approximately ± 0.005. The corresponding results are presented in Tables 2 to 13 (see Appendix).

From the empirical evidence in Tables 2 to 13, it can be seen that the power of the proposed test has an increasing pattern with respect to m (i.e., the number of failures observed) under any null and alternative distributions. This reveals the consistency of the proposed test, while this does not hold for the tests D, W, and A. In general, we observe that the proposed test performs very well for all null and alternative distributions considered, with only a few exceptions. Furthermore, Tables 2-13 show that, across all alternative distributions, the proposed test exhibits higher power than the competing tests D, W, and A. Note that for computing the test statistics D, W, and A under the assumption of a Pareto null distribution, the maximum likelihood estimate of θ is given by

$$\hat{\theta} = \frac{m}{\sum_{i=1}^{m}(R_i + 1)\log X_{i:m:n}}.$$

In our simulation study, the estimated values of $\hat{\theta}$ frequently took negative values when the data were generated from positive-valued alternative distributions such as the Weibull, gamma, and log-normal distributions. Since the shape parameter θ of the Pareto distribution must be strictly positive, this leads to infeasibility in computing $X^{\star}_{i:m:n}$. Consequently, the test statistics proposed by Pakyari and Balakrishnan (2012) cannot be computed when the data are generated from such positive-valued alternatives. For this reason, when the null hypothesis is the Pareto distribution, we included alternative distributions such as the log-gamma and log-Weibull distributions in the power simulation study.

4. Illustrative examples

In this section, we present some real-world examples to illustrate the application of the proposed GOF test to assess the validity of the null distribution when a progressively first-failure-censored sample is available.

Example 4.1. Exponential distribution

The following data are from Lawless (2003) and consist of failure times for 36 appliances subjected to an automatic life test.

11	35	49	170	329	381	708	958	1062
1167	1594	1925	1990	2223	2327	2400	2451	2471
2551	2565	2568	2694	2702	2761	2831	3034	3059
3112	3214	3478	3504	4329	6367	6976	7846	13403

Assuming $k = 3$, we randomly generate a progressively first-failure-censored sample of size $m = 8$ from the observations above. The observed failure times and the progressive censoring pattern employed are reported below:

i	1	2	3	4	5	6	7	8
$X_{i:m:n:k}$	11	170	329	958	1062	1167	1990	2223
R_i	0	0	0	0	0	1	1	2

The test statistic is $T = 19.52794$. Setting $\alpha = 0.05$, since $5.62873 = \chi^2_{0.025,14} < T < \chi^2_{0.975,14} = 26.11895$, we fail to reject the null hypothesis that the random sample comes from an exponential distribution. This result is consistent with the findings of Baratpour and Habibi Rad (2012) and Sadeghpour *et al.* (2018).

Example 4.2. Weibull distribution
We consider the following data, which are the failure times of 25 ball bearings in an endurance test as presented in Caroni (2002).

17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84	51.96	54.12
55.56	67.80	67.80	67.80	68.64	68.64	68.88	84.12	93.12	98.64
105.12	105.84	127.92	128.04	173.40					

We generated a progressively first-failure-censored sample with $k = 1$, $m = 10$ from the above data. After running this censoring scheme, the following censored sample was generated.

i	1	2	3	4	5	6	7	8	9	10
$X_{i:m:n:k}$	17.88	28.92	41.52	42.12	45.60	48.48	67.80	67.80	84.12	93.12
R_i	0	1	2	2	2	2	2	2	1	1

Here, we use the proposed test to assess whether the censored data come from a Weibull distribution with DF $F(x) = 1 - \exp\left\{-\theta x^{\beta}\right\}$ for $x > 0$ with shape parameter $\beta = 3$. From the obtained censored data, we compute the test statistic as $T = 14.88533$. Since $\chi^2_{0.025,18} = 8.230746 < T < \chi^2_{0.975,18} = 31.52638$, we fail to reject the null hypothesis that the failure times follow the Weibull distribution with $\beta = 3$ at the significance level $\alpha = 0.05$.

Example 4.3. Pareto distribution
In Nelson (1982), a lifetime dataset on insulating fluid tested at high voltage stress is provided. The times to breakdown (in minutes) are reported below:

A progressive first-failure censoring scheme with $k = 2$, $m = 20$ was conducted, and the observed data are presented below:

The test statistic is $T_\star = 6.8246$. Then, we have $T_\star < \chi^2_{0.025,36} = 21.33588$. This result provides very strong evidence to reject the null hypothesis that the censored sample comes from a Pareto distribution.

1.89	4.03	1.54	0.31	0.66	1.70	2.17	1.82	9.99	2.24
1.99	1.17	0.64	3.87	2.15	2.80	1.08	0.70	2.57	3.82
0.93	0.02	4.75	0.50	0.82	3.72	2.06	0.06	0.49	3.57
8.11	3.17	5.55	0.80	0.20	1.13	6.63	1.08	2.44	0.78
2.12	3.97	1.56	1.00	1.00	8.71	2.10	7.21	3.83	5.13

i	1	2	3	4	5	6	7	8	9	10
$X_{i:m:n:k}$	0.02	0.06	0.20	0.31	0.49	0.50	0.64	0.66	0.70	0.80
R_i	1	1	1	0	0	0	0	0	0	0
i	11	12	13	14	15	16	17	18	19	20
$X_{i:m:n:k}$	0.93	1.00	1.08	1.08	1.17	1.56	2.12	2.17	2.44	2.80
R_i	0	0	0	0	0	0	0	0	1	1

Conclusions

In this article, we developed a general goodness-of-fit test to assess whether a progressively first-failure censored sample originates from a distribution belonging to the PHR model. The proposed test thus accommodates a variety of null lifetime distributions. Under the null hypothesis, the test statistic follows a chi-squared distribution, allowing critical values to be obtained directly from standard chi-squared tables without the need for simulation. Moreover, these critical values depend only on the number of observations m and are independent of the censoring scheme $(R_1, \ldots, R_m)$. Also, the test proposed in this article is an omnibus GOF test, meaning it is designed to detect general departures from the hypothesized null distribution across the entire range of the distribution, rather than being specifically sensitive to deviations in the tails.

In the simulation study, we examine the power of the test under exponential, Rayleigh, and Pareto null distributions. The simulation results provide evidence of the test's good performance against various alternative distributions. Additionally, we compare the performance of our test with three tests introduced by Pakyari and Balakrishnan (2013), and the simulation results show that the proposed test performs substantially better than the competing tests.

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Appendix

Table 1: Different choices for progressive censoring scheme.

n	m	$R_1, \ldots, R_m$	CS
20	8	$R_1 = 12,\ R_i = 0$ for $i \neq 1$	1
		$R_8 = 12,\ R_i = 0$ for $i \neq 8$	2
		$R_1 = R_8 = 6,\ R_i = 0$ for $i \neq 1, 8$	3
	12	$R_1 = 8,\ R_i = 0$ for $i \neq 1$	4
		$R_{12} = 8,\ R_i = 0$ for $i \neq 12$	5
		$R_3 = R_5 = R_7 = R_9 = 2,\ R_i = 0$ for $i \neq 3, 5, 7, 9$	6
	16	$R_1 = 4,\ R_i = 0$ for $i \neq 1$	7
		$R_{16} = 4,\ R_i = 0$ for $i \neq 16$	8
		$R_5 = 4,\ R_i = 0$ for $i \neq 5$	9
40	10	$R_1 = 30,\ R_i = 0$ for $i \neq 1$	10
		$R_{10} = 30,\ R_i = 0$ for $i \neq 10$	11
		$R_1 = R_5 = R_{10} = 10,\ R_i = 0$ for $i \neq 1, 5, 10$	12
	20	$R_1 = 20,\ R_i = 0$ for $i \neq 1$	13
		$R_{20} = 20,\ R_i = 0$ for $i \neq 20$	14
		$R_i = 1$ for $i = 1, 2, \ldots, 20$	15
	30	$R_1 = 10,\ R_i = 0$ for $i \neq 1$	16
		$R_{30} = 10,\ R_i = 0$ for $i \neq 30$	17
		$R_1 = R_{30} = 5,\ R_i = 0$ for $i \neq 1, 30$	18
60	20	$R_1 = 40,\ R_i = 0$ for $i \neq 1$	19
		$R_{20} = 40,\ R_i = 0$ for $i \neq 20$	20
		$R_1 = R_{20} = 10,\ R_{10} = 20,\ R_i = 0$ for $i \neq 1, 10, 20$	21
	40	$R_1 = 20,\ R_i = 0$ for $i \neq 1$	22
		$R_{40} = 20,\ R_i = 0$ for $i \neq 40$	23
		$R_{2i-1} = 1,\ R_{2i} = 0,$ for $i = 1, 2, \ldots, 20$	24
	50	$R_1 = 10,\ R_i = 0$ for $i \neq 1$	25
		$R_{50} = 10,\ R_i = 0$ for $i \neq 50$	26
		$R_1 = R_{50} = 5,\ R_i = 0$ for $i \neq 1, 50$	27

Table 2: Power of the proposed and competing tests for alternatives W(0.5), W(1.5) and LN(0.5) at $\alpha = 0.1$ when the null hypothesis is the exponential distribution and $k = 2$.

CS	W(0.5)				W(1.5)				LN(0.5)			
	T	D	W	A	T	D	W	A	T	D	W	A
1	0.8690	0.2071	0.2193	0.2155	0.4294	0.0701	0.0712	0.0677	0.9994	0.1567	0.1658	0.1631
2	0.6463	0.0838	0.0855	0.0843	0.2453	0.1083	0.1112	0.1098	0.9918	0.0973	0.0963	0.0941
3	0.7418	0.0860	0.0841	0.0815	0.2991	0.1140	0.1196	0.1169	0.9974	0.0955	0.0947	0.0895
4	0.9347	0.2410	0.2497	0.2518	0.5292	0.0580	0.0588	0.0567	1.0000	0.1781	0.1826	0.1819
5	0.8295	0.0914	0.0985	0.0943	0.3714	0.1061	0.0997	0.1031	0.9999	0.0912	0.0935	0.0899
6	0.9165	0.2483	0.2524	0.2570	0.4630	0.0546	0.0561	0.0559	1.0000	0.1668	0.1680	0.1669
7	0.9611	0.2601	0.2760	0.2812	0.6048	0.0472	0.0519	0.0505	1.0000	0.1820	0.1909	0.1901
8	0.9238	0.1427	0.1478	0.1335	0.5013	0.0909	0.0954	0.1051	1.0000	0.1240	0.1214	0.1173
9	0.9648	0.2777	0.2776	0.2823	0.5932	0.0554	0.0514	0.0522	1.0000	0.1933	0.1940	0.1974
10	0.9471	0.2444	0.2461	0.2481	0.5808	0.0530	0.0513	0.0487	1.0000	0.1778	0.1831	0.1752
11	0.7249	0.0856	0.0826	0.0844	0.2950	0.1146	0.1147	0.1134	0.9998	0.0964	0.0978	0.0939
12	0.8281	0.1016	0.1110	0.1121	0.3706	0.1013	0.0997	0.1027	1.0000	0.0990	0.1069	0.1090
13	0.9926	0.3327	0.3422	0.3481	0.7577	0.0480	0.0492	0.0469	1.0000	0.2125	0.2157	0.2216
14	0.9425	0.0952	0.1047	0.1022	0.5525	0.1179	0.1178	0.1227	1.0000	0.0927	0.0940	0.0955
15	0.9797	0.2504	0.2550	0.2530	0.6547	0.0514	0.0500	0.0492	1.0000	0.1609	0.1625	0.1616
16	0.9981	0.3650	0.3929	0.3852	0.8661	0.0375	0.0423	0.0347	1.0000	0.2239	0.2467	0.2401
17	0.9929	0.1392	0.1627	0.1503	0.7731	0.1012	0.0990	0.1130	1.0000	0.1119	0.1246	0.1177
18	0.9966	0.2151	0.2152	0.2143	0.8121	0.0817	0.0893	0.0999	1.0000	0.1594	0.1560	0.1591
19	0.9944	0.3381	0.3464	0.3558	0.7995	0.0397	0.0377	0.0379	1.0000	0.2178	0.2169	0.2218
20	0.9323	0.0906	0.0858	0.0874	0.5321	0.1116	0.1092	0.1070	1.0000	0.0899	0.0837	0.0834
21	0.9811	0.1336	0.1328	0.1350	0.6547	0.0854	0.0853	0.0892	1.0000	0.1196	0.1246	0.1243
22	1.0000	0.4182	0.4469	0.4481	0.9470	0.0331	0.0409	0.0282	1.0000	0.2586	0.2776	0.2802
23	0.9987	0.1231	0.1445	0.1440	0.8707	0.1294	0.1227	0.1319	1.0000	0.1054	0.1125	0.1089
24	0.9999	0.3494	0.3918	0.3705	0.9227	0.0394	0.0349	0.0381	1.0000	0.2053	0.2260	0.2156
25	0.9999	0.4769	0.4980	0.5092	0.9752	0.0346	0.0426	0.0251	1.0000	0.2785	0.2918	0.2983
26	0.9998	0.2332	0.2415	0.2360	0.9473	0.0958	0.0947	0.1099	1.0000	0.1541	0.1543	0.1512
27	0.9999	0.3612	0.3372	0.3497	0.9591	0.0749	0.0844	0.0875	1.0000	0.2238	0.2060	0.2126

Table 3: Power of the proposed and competing tests for alternatives LN(1), G(0.75) and G(2) at $\alpha = 0.1$ when the null hypothesis is the exponential distribution and $k = 2$.

CS	LN(1)				G(0.75)				G(2)			
	T	D	W	A	T	D	W	A	T	D	W	A
1	0.4546	0.2258	0.2362	0.2340	0.2745	0.1119	0.1195	0.1168	0.5908	0.0940	0.1013	0.0956
2	0.4423	0.0843	0.0865	0.0833	0.2216	0.0941	0.0910	0.0908	0.3934	0.1010	0.1003	0.1022
3	0.4719	0.0900	0.0871	0.0840	0.2383	0.1003	0.1032	0.1004	0.4579	0.1027	0.1047	0.1019
4	0.4814	0.2758	0.2897	0.2838	0.2960	0.1068	0.1052	0.1065	0.6890	0.0922	0.0893	0.0900
5	0.5585	0.0923	0.0983	0.0943	0.2621	0.0986	0.0972	0.0897	0.5537	0.0998	0.1003	0.1007
6	0.5221	0.2489	0.2601	0.2596	0.3091	0.1035	0.1065	0.1050	0.6354	0.0850	0.0836	0.0858
7	0.4893	0.2874	0.3196	0.3098	0.3335	0.1061	0.1110	0.1117	0.7586	0.0843	0.0865	0.0879
8	0.5628	0.1541	0.1616	0.1497	0.3137	0.1050	0.1005	0.1006	0.6993	0.0959	0.0957	0.1030
9	0.4982	0.2993	0.3119	0.3105	0.3346	0.1113	0.1079	0.1099	0.7593	0.0974	0.0937	0.0957
10	0.7061	0.2697	0.2850	0.2783	0.3172	0.1077	0.1108	0.1056	0.7937	0.0873	0.0883	0.0840
11	0.6655	0.0885	0.0848	0.0869	0.2530	0.0980	0.0967	0.0907	0.4950	0.1085	0.1044	0.1002
12	0.7412	0.1106	0.1202	0.1192	0.2833	0.1013	0.1109	0.1099	0.6033	0.1014	0.1033	0.1030
13	0.7406	0.3530	0.3825	0.3722	0.4067	0.1171	0.1097	0.1127	0.9101	0.0905	0.0890	0.0885
14	0.8636	0.0869	0.0940	0.1001	0.3591	0.0922	0.0954	0.0946	0.8097	0.1039	0.1066	0.1095
15	0.8549	0.2359	0.2421	0.2403	0.4138	0.1119	0.1084	0.1082	0.8871	0.0895	0.0868	0.0867
16	0.7581	0.4101	0.4577	0.4395	0.4851	0.1040	0.1038	0.1011	0.9662	0.0853	0.0899	0.0854
17	0.8830	0.1515	0.1786	0.1728	0.4524	0.0995	0.1073	0.1056	0.9387	0.0988	0.1002	0.0986
18	0.8571	0.2466	0.2515	0.2540	0.4628	0.1037	0.1057	0.1056	0.9487	0.0953	0.0967	0.0991
19	0.8531	0.3702	0.3984	0.3948	0.4313	0.1070	0.1019	0.1065	0.9381	0.0864	0.0835	0.0862
20	0.9348	0.0917	0.0902	0.0902	0.3596	0.0962	0.0952	0.0923	0.8221	0.1084	0.1030	0.1031
21	0.9544	0.1400	0.1424	0.1402	0.4263	0.1087	0.1040	0.1055	0.8985	0.0997	0.1005	0.1003
22	0.8852	0.4736	0.5325	0.5125	0.5773	0.1099	0.1114	0.1108	0.9927	0.0850	0.0821	0.0842
23	0.9794	0.1280	0.1540	0.1514	0.5477	0.1002	0.0960	0.0950	0.9844	0.1034	0.0970	0.0993
24	0.9563	0.3355	0.3887	0.3586	0.5946	0.1145	0.1178	0.1157	0.9932	0.0836	0.0827	0.0848
25	0.8941	0.5239	0.5717	0.5613	0.6362	0.1152	0.1084	0.1179	0.9979	0.0891	0.0869	0.0872
26	0.9729	0.2701	0.2733	0.2773	0.6088	0.1055	0.1031	0.1035	0.9964	0.0969	0.0935	0.0926
27	0.9564	0.4028	0.3809	0.3999	0.6215	0.1167	0.1121	0.1129	0.9966	0.0969	0.0907	0.0950

Table 4: Power of the proposed and competing tests for alternatives W(0.5), W(1.5) and LN(0.5) at $\alpha = 0.1$ when the null hypothesis is the exponential distribution and $k = 3$.

CS	W(0.5)				W(1.5)				LN(0.5)			
	T	D	W	A	T	D	W	A	T	D	W	A
1	0.8674	0.1945	0.1965	0.1964	0.4325	0.0658	0.0658	0.0654	1.0000	0.1411	0.1422	0.1417
2	0.6513	0.0849	0.0875	0.0881	0.2469	0.1159	0.1189	0.1149	0.9964	0.0955	0.0955	0.0922
3	0.7356	0.0854	0.0830	0.0859	0.2956	0.1127	0.1089	0.1144	0.9997	0.0914	0.0900	0.0921
4	0.9314	0.2351	0.2438	0.2443	0.5216	0.0578	0.0556	0.0566	1.0000	0.1574	0.1643	0.1613
5	0.8178	0.1005	0.1095	0.1025	0.3693	0.1116	0.1085	0.1087	1.0000	0.0993	0.0985	0.0937
6	0.9171	0.2349	0.2457	0.2439	0.4596	0.0572	0.0595	0.0590	1.0000	0.1480	0.1583	0.1568
7	0.9614	0.2560	0.2709	0.2673	0.5923	0.0545	0.0537	0.0538	1.0000	0.1660	0.1733	0.1710
8	0.9241	0.1609	0.1649	0.1478	0.5053	0.0857	0.0908	0.1007	1.0000	0.1281	0.1358	0.1247
9	0.9629	0.2509	0.2631	0.2595	0.5941	0.0587	0.0558	0.0566	1.0000	0.1767	0.1790	0.1782
10	0.9434	0.2382	0.2425	0.2436	0.5847	0.0618	0.0597	0.0596	1.0000	0.1584	0.1573	0.1597
11	0.7213	0.0936	0.0910	0.0943	0.2887	0.1134	0.1116	0.1110	1.0000	0.1003	0.0925	0.0961
12	0.8377	0.0951	0.1010	0.1073	0.3658	0.0941	0.0970	0.0995	1.0000	0.0958	0.1003	0.1041
13	0.9927	0.2884	0.3145	0.3103	0.7513	0.0436	0.0464	0.0455	1.0000	0.1899	0.2020	0.1970
14	0.9432	0.0867	0.0972	0.1024	0.5481	0.1067	0.1066	0.1140	1.0000	0.0944	0.0931	0.0957
15	0.9829	0.2391	0.2491	0.2527	0.6665	0.0586	0.0598	0.0603	1.0000	0.1522	0.1588	0.1603
16	0.9989	0.3454	0.3679	0.3633	0.8592	0.0418	0.0354	0.0360	1.0000	0.2084	0.2212	0.2169
17	0.9936	0.1603	0.1764	0.1633	0.7695	0.1004	0.1045	0.1167	1.0000	0.1194	0.1249	0.1174
18	0.9961	0.2489	0.2482	0.2240	0.8118	0.0685	0.0739	0.0872	1.0000	0.1622	0.1624	0.1508
19	0.9948	0.3044	0.3215	0.3188	0.8070	0.0469	0.0423	0.0441	1.0000	0.1915	0.2005	0.1996
20	0.9337	0.0909	0.0916	0.0978	0.5302	0.1112	0.1132	0.1164	1.0000	0.0979	0.0981	0.0977
21	0.9745	0.1390	0.1324	0.1281	0.6539	0.0913	0.0874	0.0929	1.0000	0.1202	0.1173	0.1114
22	0.9998	0.3951	0.4367	0.4180	0.9456	0.0374	0.0323	0.0332	1.0000	0.2294	0.2506	0.2403
23	0.9986	0.1304	0.1549	0.1434	0.8679	0.1107	0.1091	0.1182	1.0000	0.1107	0.1178	0.1149
24	0.9997	0.3178	0.3425	0.3359	0.9280	0.0425	0.0403	0.0404	1.0000	0.1827	0.1911	0.1882
25	1.0000	0.4144	0.4717	0.4518	0.9706	0.0360	0.0315	0.0326	1.0000	0.2325	0.2594	0.2466
26	0.9998	0.2505	0.2503	0.2343	0.9433	0.0785	0.0791	0.0973	1.0000	0.1612	0.1609	0.1470
27	1.0000	0.3465	0.3466	0.3289	0.9617	0.0493	0.0559	0.0685	1.0000	0.2112	0.2085	0.1966

Table 5: Power of the proposed and competing tests for alternatives LN(1), G(0.75) and G(2) at $\alpha = 0.1$ when the null hypothesis is the exponential distribution and $k = 3$.

CS	LN(1)				G(0.75)				G(2)			
	T	D	W	A	T	D	W	A	T	D	W	A
1	0.6268	0.1906	0.1961	0.1962	0.2822	0.1124	0.1114	0.1101	0.6215	0.0833	0.0826	0.0822
2	0.5362	0.0889	0.0897	0.0869	0.2318	0.0979	0.0946	0.0956	0.4120	0.1037	0.1054	0.1047
3	0.5935	0.0948	0.0898	0.0941	0.2509	0.1009	0.0931	0.0962	0.4827	0.1076	0.1067	0.1115
4	0.6635	0.2338	0.2461	0.2419	0.3145	0.1109	0.1125	0.1127	0.7381	0.0903	0.0887	0.0884
5	0.6822	0.0999	0.1039	0.0951	0.2657	0.1034	0.1034	0.1016	0.5982	0.1121	0.1089	0.1104
6	0.6878	0.2192	0.2302	0.2271	0.3313	0.1093	0.1082	0.1089	0.6767	0.0877	0.0914	0.0897
7	0.7033	0.2524	0.2703	0.2663	0.3549	0.1125	0.1147	0.1146	0.7992	0.0931	0.0922	0.0897
8	0.7446	0.1651	0.1717	0.1531	0.3403	0.1081	0.1120	0.1099	0.7431	0.0949	0.0994	0.1027
9	0.7183	0.2610	0.2730	0.2707	0.3734	0.1150	0.1158	0.1154	0.8042	0.0910	0.0867	0.0878
10	0.8549	0.2322	0.2449	0.2456	0.3417	0.1231	0.1194	0.1194	0.8195	0.0847	0.0830	0.0835
11	0.7617	0.0955	0.0917	0.0903	0.2560	0.1005	0.0937	0.0972	0.5172	0.1160	0.1123	0.1101
12	0.8435	0.0939	0.1025	0.1064	0.2932	0.0910	0.0968	0.0990	0.6279	0.0939	0.0956	0.0973
13	0.9138	0.2809	0.3139	0.3081	0.4405	0.1121	0.1149	0.1138	0.9375	0.0754	0.0789	0.0780
14	0.9441	0.0949	0.1024	0.1035	0.3718	0.0939	0.0997	0.1022	0.8402	0.1057	0.1025	0.1084
15	0.9533	0.2160	0.2244	0.2280	0.4435	0.1155	0.1177	0.1195	0.9069	0.0843	0.0833	0.0844
16	0.9475	0.3432	0.3731	0.3659	0.5203	0.1180	0.1160	0.1178	0.9780	0.0849	0.0827	0.0833
17	0.9750	0.1526	0.1722	0.1602	0.4845	0.1009	0.1041	0.1021	0.9580	0.0968	0.1010	0.1071
18	0.9674	0.2581	0.2554	0.2365	0.4945	0.1152	0.1152	0.1131	0.9704	0.0848	0.0915	0.0932
19	0.9644	0.3103	0.3378	0.3316	0.4664	0.1173	0.1188	0.1185	0.9576	0.0827	0.0804	0.0808
20	0.9752	0.0982	0.0991	0.1027	0.3736	0.1001	0.0984	0.1024	0.8483	0.1062	0.1062	0.1106
21	0.9871	0.1434	0.1386	0.1342	0.4471	0.1069	0.1005	0.0993	0.9200	0.0951	0.0930	0.0926
22	0.9858	0.3955	0.4464	0.4249	0.6266	0.1246	0.1283	0.1248	0.9975	0.0851	0.0830	0.0810
23	0.9975	0.1279	0.1566	0.1491	0.5749	0.1035	0.1066	0.1027	0.9920	0.1043	0.1054	0.1085
24	0.9979	0.2762	0.2991	0.2933	0.6302	0.1197	0.1201	0.1187	0.9965	0.0798	0.0776	0.0782
25	0.9926	0.4076	0.4652	0.4439	0.6832	0.1108	0.1192	0.1141	0.9991	0.0785	0.0763	0.0779
26	0.9996	0.2628	0.2560	0.2410	0.6494	0.1049	0.1002	0.0983	0.9987	0.0857	0.0834	0.0876
27	0.9970	0.3517	0.3533	0.3405	0.6711	0.1169	0.1177	0.1120	0.9995	0.0789	0.0822	0.0819

Table 6: Power of the proposed and competing tests for alternatives W(0.5), W(1.5) and LN(0.5) at $\alpha = 0.1$ when the null hypothesis is the Rayleigh distribution and $k = 2$.

CS	W(0.5)				W(1.5)				LN(0.5)			
	T	D	W	A	T	D	W	A	T	D	W	A
1	0.9990	0.3517	0.3462	0.3575	0.3441	0.1426	0.1417	0.1436	0.4665	0.2329	0.2345	0.2435
2	0.9792	0.0736	0.0771	0.0833	0.2452	0.0944	0.0916	0.0911	0.4532	0.0905	0.0908	0.0912
3	0.9907	0.0642	0.0788	0.0843	0.2702	0.0842	0.0894	0.0929	0.4775	0.0765	0.0809	0.0790
4	0.9998	0.4313	0.4397	0.4465	0.4082	0.1513	0.1503	0.1494	0.4745	0.2543	0.2692	0.2672
5	0.9987	0.0953	0.1261	0.1167	0.3152	0.0950	0.0928	0.0902	0.5333	0.0928	0.1005	0.0915
6	1.0000	0.4349	0.4341	0.4493	0.4035	0.1488	0.1478	0.1518	0.5104	0.2584	0.2643	0.2691
7	1.0000	0.5075	0.5299	0.5305	0.4605	0.1616	0.1624	0.1605	0.4910	0.3012	0.3216	0.3109
8	1.0000	0.2476	0.2498	0.2331	0.4016	0.1116	0.1118	0.1080	0.5578	0.1538	0.1574	0.1462
9	1.0000	0.4967	0.5132	0.5218	0.4738	0.1503	0.1550	0.1539	0.5026	0.3046	0.3170	0.3209
10	1.0000	0.4223	0.4289	0.4319	0.4213	0.1526	0.1575	0.1558	0.6982	0.2685	0.2826	0.2775
11	0.9902	0.0808	0.0818	0.0853	0.2705	0.0861	0.0895	0.0907	0.6784	0.0807	0.0870	0.0831
12	0.9975	0.0934	0.1148	0.1257	0.3166	0.0946	0.1046	0.1065	0.7577	0.1056	0.1168	0.1177
13	1.0000	0.6015	0.6363	0.6343	0.5571	0.1738	0.1726	0.1799	0.7349	0.3518	0.3954	0.3831
14	1.0000	0.1092	0.1395	0.1432	0.4226	0.0877	0.0908	0.0901	0.8650	0.0890	0.0996	0.1005
15	1.0000	0.4841	0.5157	0.5111	0.5118	0.1538	0.1573	0.1568	0.8556	0.2555	0.2714	0.2679
16	1.0000	0.7067	0.7433	0.7452	0.6700	0.1902	0.1806	0.1951	0.7674	0.4176	0.4527	0.4494
17	1.0000	0.2569	0.3214	0.3149	0.5819	0.1026	0.1101	0.1086	0.8866	0.1387	0.1695	0.1677
18	1.0000	0.4523	0.4508	0.4495	0.6060	0.1298	0.1317	0.1285	0.8591	0.2336	0.2443	0.2453
19	1.0000	0.6368	0.6579	0.6632	0.5864	0.1911	0.1919	0.1916	0.8488	0.4036	0.4239	0.4212
20	1.0000	0.1176	0.1267	0.1316	0.4165	0.0882	0.0891	0.0904	0.9383	0.0906	0.0923	0.0909
21	1.0000	0.1960	0.1975	0.2024	0.5092	0.1084	0.1144	0.1120	0.9502	0.1368	0.1400	0.1387
22	1.0000	0.8153	0.8549	0.8495	0.7811	0.2085	0.2147	0.2172	0.8895	0.4894	0.5464	0.5262
23	1.0000	0.2415	0.3411	0.3418	0.6651	0.0903	0.1008	0.1029	0.9785	0.1190	0.1525	0.1574
24	1.0000	0.6909	0.7715	0.7279	0.7534	0.1865	0.2003	0.1927	0.9554	0.3624	0.4218	0.3911
25	1.0000	0.8510	0.8944	0.8921	0.8402	0.2252	0.2264	0.2399	0.8992	0.5240	0.5904	0.5796
26	1.0000	0.5576	0.5812	0.5827	0.7748	0.1319	0.1343	0.1326	0.9738	0.2628	0.2852	0.2847
27	1.0000	0.7260	0.7115	0.7269	0.8025	0.1817	0.1700	0.1711	0.9542	0.4121	0.4007	0.4107

Table 7: Power of the proposed and competing tests for alternatives LN(1), G(0.75) and G(2) at $\alpha = 0.1$ when the null hypothesis is the Rayleigh distribution and $k = 2$.

CS	LN(1)				G(0.75)				G(2)			
	T	D	W	A	T	D	W	A	T	D	W	A
1	0.3944	0.3569	0.3766	0.3780	0.9662	0.2306	0.2275	0.2321	0.2246	0.1891	0.1868	0.1903
2	0.0780	0.0809	0.0854	0.0894	0.8550	0.0755	0.0837	0.0875	0.1263	0.0871	0.0926	0.0955
3	0.1176	0.0682	0.0775	0.0807	0.9016	0.0687	0.0747	0.0811	0.1404	0.0845	0.0864	0.0867
4	0.5401	0.4296	0.4627	0.4578	0.9909	0.2581	0.2611	0.2657	0.2713	0.2001	0.2095	0.2054
5	0.1610	0.0917	0.1195	0.1112	0.9494	0.0880	0.1008	0.0937	0.1676	0.0923	0.0980	0.0940
6	0.4381	0.4231	0.4436	0.4411	0.9839	0.2703	0.2718	0.2767	0.2479	0.2005	0.2025	0.2049
7	0.6543	0.5054	0.5470	0.5314	0.9977	0.2994	0.3121	0.3132	0.3258	0.2291	0.2344	0.2366
8	0.3661	0.2255	0.2423	0.2300	0.9895	0.1540	0.1555	0.1443	0.2355	0.1306	0.1337	0.1271
9	0.6489	0.4913	0.5315	0.5184	0.9970	0.2968	0.3071	0.3137	0.3294	0.2152	0.2194	0.2233
10	0.4407	0.4312	0.4744	0.4639	0.9939	0.2608	0.2630	0.2683	0.2573	0.2061	0.2127	0.2101
11	0.0655	0.0777	0.0809	0.0848	0.9055	0.0832	0.0860	0.0883	0.1179	0.0917	0.0945	0.0906
12	0.0943	0.1083	0.1188	0.1251	0.9610	0.0946	0.1068	0.1134	0.1513	0.0953	0.1045	0.1091
13	0.7286	0.5900	0.6638	0.6416	0.9997	0.3554	0.3732	0.3781	0.3843	0.2571	0.2701	0.2729
14	0.1386	0.1035	0.1278	0.1334	0.9948	0.0922	0.1045	0.1066	0.1869	0.0860	0.0915	0.0921
15	0.3735	0.4007	0.4322	0.4283	0.9988	0.3053	0.3185	0.3160	0.2709	0.1965	0.2033	0.2017
16	0.8763	0.6809	0.7461	0.7316	1.0000	0.4093	0.4193	0.4300	0.4880	0.3004	0.3089	0.3152
17	0.4874	0.2388	0.2961	0.2952	0.9997	0.1423	0.1758	0.1719	0.3202	0.1186	0.1315	0.1293
18	0.6503	0.4372	0.4601	0.4685	0.9998	0.2373	0.2375	0.2365	0.3780	0.1771	0.1802	0.1775
19	0.7123	0.6413	0.7034	0.6878	0.9999	0.3802	0.3869	0.3918	0.3930	0.2862	0.2952	0.2973
20	0.0799	0.0971	0.1046	0.1079	0.9944	0.0918	0.0958	0.0991	0.1579	0.0960	0.0931	0.0962
21	0.1788	0.1998	0.1986	0.1979	0.9985	0.1371	0.1418	0.1367	0.2216	0.1215	0.1210	0.1171
22	0.9372	0.7846	0.8621	0.8329	1.0000	0.4825	0.5104	0.5110	0.6002	0.3473	0.3705	0.3646
23	0.4401	0.2045	0.2922	0.3066	0.9999	0.1258	0.1706	0.1695	0.3357	0.0993	0.1269	0.1247
24	0.8068	0.6111	0.6976	0.6539	1.0000	0.4219	0.4725	0.4459	0.4844	0.2742	0.3058	0.2859
25	0.9749	0.8280	0.9008	0.8821	1.0000	0.5249	0.5634	0.5778	0.6702	0.3685	0.3960	0.4059
26	0.8038	0.5131	0.5480	0.5709	1.0000	0.2687	0.2830	0.2818	0.4879	0.1903	0.2007	0.1978
27	0.8935	0.6996	0.7146	0.7344	1.0000	0.4009	0.3824	0.3934	0.5729	0.2636	0.2517	0.2588

Table 8: Power of the proposed and competing tests for alternatives W(0.5), W(1.5) and LN(0.5) at $\alpha = 0.1$ when the null hypothesis is the Rayleigh distribution and $k = 3$.

CS	W(0.5)				W(1.5)				LN(0.5)			
	T	D	W	A	T	D	W	A	T	D	W	A
1	0.9991	0.3189	0.3193	0.3186	0.3428	0.1303	0.1298	0.1305	0.6208	0.2030	0.2026	0.2054
2	0.9787	0.0788	0.0814	0.0832	0.2445	0.0933	0.0949	0.0990	0.5466	0.0902	0.0896	0.0897
3	0.9912	0.0753	0.0808	0.0864	0.2758	0.0900	0.0950	0.0981	0.5920	0.0880	0.0905	0.0901
4	0.9999	0.3962	0.4137	0.4099	0.4086	0.1474	0.1491	0.1471	0.6703	0.2246	0.2310	0.2289
5	0.9983	0.1010	0.1279	0.1224	0.3174	0.0980	0.0994	0.0982	0.6785	0.0961	0.1045	0.0981
6	0.9996	0.4014	0.4144	0.4151	0.4063	0.1455	0.1491	0.1511	0.6826	0.2186	0.2261	0.2255
7	1.0000	0.4662	0.4994	0.4916	0.4620	0.1471	0.1527	0.1493	0.7116	0.2486	0.2697	0.2646
8	1.0000	0.2705	0.2547	0.2245	0.3953	0.1189	0.1135	0.1066	0.7480	0.1688	0.1570	0.1418
9	1.0000	0.4561	0.4895	0.4832	0.4754	0.1520	0.1601	0.1585	0.7257	0.2516	0.2762	0.2720
10	0.9999	0.3821	0.3963	0.3942	0.4163	0.1462	0.1516	0.1512	0.8590	0.2202	0.2279	0.2269
11	0.9908	0.0840	0.0804	0.0873	0.2625	0.1027	0.0980	0.0946	0.7598	0.0991	0.0933	0.0954
12	0.9980	0.0954	0.1063	0.1174	0.3227	0.1004	0.0991	0.0998	0.8399	0.1050	0.1086	0.1094
13	1.0000	0.5548	0.5955	0.5882	0.5710	0.1776	0.1851	0.1842	0.9141	0.3061	0.3270	0.3235
14	1.0000	0.1254	0.1471	0.1477	0.4152	0.1055	0.1026	0.1018	0.9446	0.1042	0.1130	0.1056
15	1.0000	0.4491	0.4642	0.4658	0.5093	0.1447	0.1471	0.1470	0.9541	0.2115	0.2177	0.2175
16	1.0000	0.6548	0.7085	0.6919	0.6767	0.1798	0.1871	0.1833	0.9445	0.3416	0.3791	0.3664
17	1.0000	0.2880	0.3250	0.3003	0.5741	0.1132	0.1164	0.1051	0.9722	0.1507	0.1640	0.1515
18	1.0000	0.4787	0.4631	0.4390	0.6141	0.1401	0.1368	0.1333	0.9672	0.2384	0.2396	0.2234
19	1.0000	0.5546	0.5995	0.5885	0.5903	0.1608	0.1689	0.1671	0.9624	0.3002	0.3321	0.3241
20	0.9999	0.1209	0.1281	0.1289	0.4024	0.0921	0.0927	0.0912	0.9756	0.0899	0.0911	0.0905
21	1.0000	0.2070	0.1965	0.1935	0.5046	0.1148	0.1120	0.1116	0.9869	0.1401	0.1403	0.1362
22	1.0000	0.7222	0.7862	0.7674	0.7768	0.1847	0.1977	0.1942	0.9865	0.3794	0.4315	0.4127
23	1.0000	0.2402	0.3207	0.3186	0.6594	0.1097	0.1104	0.1098	0.9984	0.1246	0.1555	0.1505
24	1.0000	0.6083	0.6577	0.6447	0.7571	0.1752	0.1786	0.1781	0.9964	0.2832	0.2990	0.2927
25	1.0000	0.7833	0.8513	0.8278	0.8392	0.1882	0.2038	0.1987	0.9928	0.4056	0.4731	0.4488
26	1.0000	0.5737	0.5652	0.5389	0.7720	0.1487	0.1536	0.1404	0.9989	0.2600	0.2566	0.2378
27	1.0000	0.7208	0.7207	0.7064	0.7955	0.1801	0.1769	0.1696	0.9981	0.3762	0.3728	0.3619

Table 9: Power of the proposed and competing tests for alternatives LN(1), G(0.75) and G(2) at $\alpha = 0.1$ when the null hypothesis is the Rayleigh distribution and $k = 3$.

CS	LN(1)				G(0.75)				G(2)			
	T	D	W	A	T	D	W	A	T	D	W	A
1	0.2454	0.2970	0.3055	0.3077	0.9698	0.2098	0.2098	0.2090	0.1972	0.1627	0.1610	0.1617
2	0.0697	0.0819	0.0827	0.0807	0.8560	0.0844	0.0832	0.0849	0.1163	0.0963	0.0905	0.0928
3	0.0855	0.0796	0.0865	0.0898	0.9076	0.0783	0.0847	0.0847	0.1393	0.0910	0.0942	0.0956
4	0.3533	0.3648	0.3922	0.3890	0.9895	0.2486	0.2576	0.2543	0.2313	0.1901	0.1955	0.1935
5	0.1061	0.0971	0.1214	0.1112	0.9580	0.0914	0.1031	0.0964	0.1536	0.0940	0.0995	0.0947
6	0.2810	0.3562	0.3712	0.3711	0.9883	0.2611	0.2661	0.2691	0.2103	0.1805	0.1851	0.1874
7	0.4492	0.4161	0.4613	0.4496	0.9953	0.2835	0.3031	0.3001	0.2830	0.2026	0.2120	0.2097
8	0.2003	0.2500	0.2339	0.2076	0.9883	0.1859	0.1782	0.1561	0.1998	0.1419	0.1298	0.1139
9	0.4376	0.4108	0.4494	0.4409	0.9964	0.2769	0.2985	0.2946	0.2724	0.1995	0.2127	0.2116
10	0.2723	0.3648	0.3882	0.3883	0.9943	0.2463	0.2556	0.2531	0.2179	0.1850	0.1924	0.1908
11	0.0752	0.0890	0.0864	0.0900	0.9111	0.0832	0.0806	0.0852	0.1113	0.0957	0.0933	0.0945
12	0.0740	0.1047	0.1177	0.1217	0.9625	0.0987	0.1042	0.1099	0.1267	0.1012	0.1056	0.1076
13	0.4942	0.5043	0.5529	0.5460	0.9996	0.3411	0.3649	0.3603	0.3184	0.2332	0.2459	0.2450
14	0.0770	0.1075	0.1240	0.1223	0.9944	0.1034	0.1161	0.1102	0.1669	0.1077	0.1065	0.1005
15	0.2083	0.3493	0.3653	0.3676	0.9992	0.2798	0.2845	0.2845	0.2272	0.1859	0.1853	0.1845
16	0.6672	0.5731	0.6426	0.6242	1.0000	0.3861	0.4238	0.4113	0.4012	0.2585	0.2761	0.2710
17	0.2514	0.2473	0.2805	0.2641	0.9998	0.1686	0.1854	0.1652	0.2618	0.1235	0.1246	0.1168
18	0.3765	0.4101	0.4161	0.3982	1.0000	0.2779	0.2707	0.2537	0.3014	0.1789	0.1765	0.1636
19	0.4671	0.5051	0.5668	0.5516	1.0000	0.3222	0.3533	0.3439	0.3129	0.2278	0.2468	0.2396
20	0.0748	0.0957	0.1012	0.1019	0.9953	0.0946	0.0931	0.0950	0.1372	0.0900	0.0896	0.0911
21	0.0961	0.1944	0.1943	0.1907	0.9993	0.1596	0.1556	0.1543	0.1879	0.1273	0.1221	0.1194
22	0.7682	0.6558	0.7388	0.7127	1.0000	0.4443	0.4908	0.4754	0.4750	0.2769	0.3090	0.2958
23	0.1941	0.1805	0.2468	0.2523	1.0000	0.1351	0.1727	0.1666	0.2820	0.1107	0.1255	0.1207
24	0.5314	0.4746	0.5230	0.5090	1.0000	0.3789	0.4008	0.3942	0.4056	0.2235	0.2320	0.2303
25	0.8537	0.6938	0.7827	0.7545	1.0000	0.4802	0.5422	0.5193	0.5549	0.2945	0.3351	0.3191
26	0.4933	0.4825	0.4825	0.4730	1.0000	0.3146	0.3068	0.2858	0.3959	0.1905	0.1945	0.1774
27	0.6517	0.6413	0.6558	0.6533	1.0000	0.4243	0.4135	0.3965	0.4468	0.2642	0.2542	0.2428

Table 10: Power of the proposed and competing tests for alternatives LN(1), LN(1.5), LG(2) and LG(0.5) at $\alpha = 0.1$ when the null hypothesis is the Pareto distribution and $k = 2$.

CS	LN(1)				LN(1.5)				LG(2)				LG(0.5)			
	$T_\star$	D	W	A	$T_\star$	D	W	A	$T_\star$	D	W	A	$T_\star$	D	W	A
1	0.6395	–	–	–	0.6364	–	–	–	0.1880	0.0966	0.0953	0.0952	0.4584	0.1326	0.1282	0.1324
2	0.4224	–	–	–	0.4213	–	–	–	0.1631	0.1104	0.1130	0.1066	0.3725	0.0935	0.0957	0.0904
3	0.4807	–	–	–	0.4934	–	–	–	0.1739	0.1038	0.1070	0.1108	0.4042	0.0885	0.0898	0.0932
4	0.7874	–	–	–	0.7933	–	–	–	0.2767	0.0903	0.0916	0.0894	0.6130	0.1208	0.1265	0.1185
5	0.6549	–	–	–	0.6543	–	–	–	0.2384	0.1056	0.1026	0.1014	0.5447	0.0925	0.0965	0.0942
6	0.7463	–	–	–	0.7605	–	–	–	0.3010	0.0869	0.0903	0.0901	0.6674	0.1367	0.1383	0.1392
7	0.8741	–	–	–	0.8666	–	–	–	0.3642	0.0926	0.0911	0.0953	0.7109	0.1293	0.1327	0.1351
8	0.8087	–	–	–	0.8067	–	–	–	0.3378	0.1000	0.1033	0.1104	0.6753	0.1193	0.1246	0.1213
9	0.8755	–	–	–	0.8798	–	–	–	0.3894	0.0874	0.0865	0.0886	0.7597	0.1298	0.1307	0.1282
10	0.8207	–	–	–	0.8211	–	–	–	0.2737	0.0949	0.0936	0.0970	0.5922	0.1392	0.1401	0.1452
11	0.5279	–	–	–	0.5209	–	–	–	0.2019	0.1089	0.1060	0.1059	0.4626	0.0910	0.0906	0.0962
12	0.6439	–	–	–	0.6427	–	–	–	0.2485	0.0982	0.0924	0.0984	0.5712	0.1012	0.1034	0.1045
13	0.9686	–	–	–	0.9670	–	–	–	0.5112	0.0862	0.0892	0.0870	0.8393	0.1392	0.1445	0.1423
14	0.8782	–	–	–	0.8802	–	–	–	0.4067	0.1010	0.0988	0.1000	0.7995	0.0928	0.0956	0.0921
15	0.9310	–	–	–	0.9280	–	–	–	0.4994	0.0850	0.0789	0.0791	0.8765	0.1544	0.1530	0.1537
16	0.9924	–	–	–	0.9919	–	–	–	0.6622	0.0863	0.0884	0.0874	0.9392	0.1344	0.1437	0.1420
17	0.9787	–	–	–	0.9787	–	–	–	0.6122	0.1007	0.1033	0.1060	0.9229	0.0982	0.1050	0.1002
18	0.9865	–	–	–	0.9849	–	–	–	0.6269	0.0958	0.0909	0.0919	0.9311	0.1288	0.1240	0.1191
19	0.9825	–	–	–	0.9812	–	–	–	0.5354	0.0868	0.0926	0.0892	0.8610	0.1445	0.1437	0.1454
20	0.8672	–	–	–	0.8719	–	–	–	0.4391	0.1049	0.1064	0.1056	0.7986	0.0838	0.0880	0.0887
21	0.9382	–	–	–	0.9321	–	–	–	0.5103	0.0946	0.0956	0.0918	0.8860	0.1192	0.1252	0.1179
22	0.9993	–	–	–	0.9998	–	–	–	0.8213	0.0903	0.0906	0.0889	0.9843	0.1569	0.1598	0.1602
23	0.9964	–	–	–	0.9965	–	–	–	0.7626	0.1090	0.1044	0.1012	0.9765	0.1020	0.1085	0.0997
24	0.9988	–	–	–	0.9986	–	–	–	0.8185	0.0818	0.0791	0.0790	0.9899	0.1521	0.1598	0.1510
25	0.9998	–	–	–	0.9999	–	–	–	0.8847	0.0841	0.0840	0.0834	0.9945	0.1494	0.1502	0.1555
26	0.9997	–	–	–	0.9995	–	–	–	0.8538	0.0937	0.0941	0.0986	0.9928	0.1316	0.1296	0.1294
27	0.9995	–	–	–	0.9995	–	–	–	0.8701	0.0924	0.0956	0.0935	0.9923	0.1443	0.1409	0.1375

Table 11: Power of the proposed and competing tests for alternatives LW(2), LW(0.5), W(2) and G(2) at $\alpha = 0.1$ when the null hypothesis is the Pareto distribution and $k = 2$.

CS	LW(2)				LW(0.5)				W(2)				G(2)			
	$T_\star$	D	W	A	$T_\star$	D	W	A	$T_\star$	D	W	A	$T_\star$	D	W	A
1	0.3314	0.0511	0.0504	0.0507	0.6376	0.2145	0.2182	0.2174	0.7869	–	–	–	0.7499	–	–	–
2	0.2157	0.1254	0.1266	0.1231	0.4186	0.0912	0.0926	0.0866	0.5316	–	–	–	0.5010	–	–	–
3	0.2526	0.1254	0.1263	0.1309	0.4708	0.0794	0.0826	0.0879	0.6150	–	–	–	0.5798	–	–	–
4	0.4915	0.0395	0.0418	0.0362	0.8056	0.2356	0.2420	0.2407	0.8914	–	–	–	0.8648	–	–	–
5	0.3543	0.1213	0.1215	0.1232	0.6531	0.0904	0.1047	0.0994	0.7624	–	–	–	0.7342	–	–	–
6	0.4574	0.0377	0.0399	0.0381	0.8195	0.2480	0.2516	0.2598	0.8562	–	–	–	0.8305	–	–	–
7	0.6055	0.0371	0.0379	0.0361	0.9009	0.2686	0.2739	0.2874	0.9469	–	–	–	0.9348	–	–	–
8	0.5167	0.0978	0.1046	0.1203	0.8325	0.1519	0.1581	0.1471	0.9064	–	–	–	0.8884	–	–	–
9	0.6138	0.0367	0.0365	0.0334	0.9141	0.2732	0.2779	0.2864	0.9447	–	–	–	0.9301	–	–	–
10	0.4768	0.0391	0.0375	0.0375	0.7637	0.2514	0.2525	0.2621	0.9174	–	–	–	0.9025	–	–	–
11	0.2653	0.1145	0.1159	0.1185	0.5108	0.0817	0.0803	0.0818	0.6395	–	–	–	0.6101	–	–	–
12	0.3343	0.0969	0.0940	0.1035	0.6366	0.0968	0.1040	0.1055	0.7551	–	–	–	0.7308	–	–	–
13	0.7718	0.0251	0.0316	0.0231	0.9617	0.3240	0.3443	0.3427	0.9931	–	–	–	0.9880	–	–	–
14	0.5762	0.1196	0.1215	0.1287	0.8661	0.0872	0.0930	0.0931	0.9433	–	–	–	0.9259	–	–	–
15	0.6950	0.0387	0.0356	0.0364	0.9433	0.2560	0.2678	0.2662	0.9754	–	–	–	0.9657	–	–	–
16	0.9008	0.0238	0.0305	0.0181	0.9927	0.3776	0.3939	0.3994	0.9988	–	–	–	0.9978	–	–	–
17	0.8217	0.1209	0.1213	0.1361	0.9820	0.1427	0.1624	0.1549	0.9955	–	–	–	0.9927	–	–	–
18	0.8597	0.0829	0.0908	0.1075	0.9856	0.2232	0.2222	0.2170	0.9972	–	–	–	0.9962	–	–	–
19	0.8144	0.0216	0.0267	0.0201	0.9645	0.3350	0.3501	0.3533	0.9961	–	–	–	0.9927	–	–	–
20	0.5560	0.1294	0.1291	0.1322	0.8506	0.0881	0.0894	0.0939	0.9339	–	–	–	0.9199	–	–	–
21	0.6832	0.0869	0.0877	0.0864	0.9344	0.1360	0.1376	0.1322	0.9751	–	–	–	0.9668	–	–	–
22	0.9703	0.0272	0.0385	0.0157	0.9998	0.4540	0.4755	0.4775	1.0000	–	–	–	0.9998	–	–	–
23	0.9096	0.1443	0.1426	0.1517	0.9956	0.1297	0.1547	0.1453	0.9992	–	–	–	0.9987	–	–	–
24	0.9557	0.0221	0.0180	0.0202	0.9994	0.3463	0.3915	0.3601	0.9999	–	–	–	0.9998	–	–	–
25	0.9862	0.0268	0.0410	0.0120	0.9999	0.4751	0.4943	0.5060	1.0000	–	–	–	1.0000	–	–	–
26	0.9741	0.1107	0.1272	0.1518	0.9992	0.2449	0.2559	0.2542	1.0000	–	–	–	0.9999	–	–	–
27	0.9811	0.0637	0.0987	0.1085	0.9996	0.3398	0.3340	0.3412	1.0000	–	–	–	1.0000	–	–	–

Table 12: Power of the proposed and competing tests for alternatives LN(1), LN(1.5), LG(2) and LG(0.5) at $\alpha = 0.1$ when the null hypothesis is the Pareto distribution and $k = 3$.

CS	LN(1)				LN(1.5)				LG(2)				LG(0.5)			
	$T_\star$	D	W	A	$T_\star$	D	W	A	$T_\star$	D	W	A	$T_\star$	D	W	A
1	0.6517	–	–	–	0.6538	–	–	–	0.2128	0.0838	0.0819	0.0824	0.5011	0.1318	0.1279	0.1288
2	0.4382	–	–	–	0.4246	–	–	–	0.1774	0.1083	0.1077	0.1044	0.3852	0.0910	0.0905	0.0876
3	0.5038	–	–	–	0.4969	–	–	–	0.1772	0.1053	0.1054	0.1092	0.4307	0.0886	0.0875	0.0868
4	0.8034	–	–	–	0.7952	–	–	–	0.3172	0.0899	0.0864	0.0879	0.6662	0.1416	0.1439	0.1434
5	0.6626	–	–	–	0.6563	–	–	–	0.2586	0.1082	0.0994	0.1040	0.5768	0.0943	0.0986	0.0954
6	0.7650	–	–	–	0.7594	–	–	–	0.3152	0.0869	0.0841	0.0842	0.7067	0.1500	0.1517	0.1500
7	0.8799	–	–	–	0.8836	–	–	–	0.3930	0.0900	0.0903	0.0886	0.7666	0.1370	0.1428	0.1393
8	0.8199	–	–	–	0.8224	–	–	–	0.3609	0.0947	0.0972	0.0973	0.7277	0.1258	0.1250	0.1141
9	0.8851	–	–	–	0.8836	–	–	–	0.4314	0.0909	0.0898	0.0901	0.8064	0.1514	0.1580	0.1549
10	0.8351	–	–	–	0.8357	–	–	–	0.3012	0.0799	0.0791	0.0806	0.6283	0.1494	0.1482	0.1504
11	0.5409	–	–	–	0.5450	–	–	–	0.2279	0.1004	0.1049	0.1053	0.4800	0.0900	0.0921	0.0906
12	0.6500	–	–	–	0.6501	–	–	–	0.2633	0.1041	0.1045	0.1091	0.5819	0.1016	0.1068	0.1105
13	0.9706	–	–	–	0.9694	–	–	–	0.5444	0.0860	0.0832	0.0840	0.8850	0.1526	0.1604	0.1600
14	0.8794	–	–	–	0.8840	–	–	–	0.4452	0.1042	0.1094	0.1102	0.8130	0.0926	0.1036	0.1009
15	0.9372	–	–	–	0.9358	–	–	–	0.5319	0.0860	0.0882	0.0881	0.8979	0.1618	0.1661	0.1668
16	0.9949	–	–	–	0.9929	–	–	–	0.7130	0.0841	0.0860	0.0848	0.9644	0.1618	0.1728	0.1657
17	0.9808	–	–	–	0.9812	–	–	–	0.6472	0.0998	0.1042	0.1093	0.9453	0.1208	0.1265	0.1160
18	0.9889	–	–	–	0.9879	–	–	–	0.6746	0.0930	0.0894	0.0927	0.9507	0.1542	0.1527	0.1474
19	0.9844	–	–	–	0.9826	–	–	–	0.5906	0.0864	0.0851	0.0850	0.8948	0.1612	0.1599	0.1621
20	0.8805	–	–	–	0.8738	–	–	–	0.4471	0.1019	0.1102	0.1131	0.8054	0.0894	0.0900	0.0935
21	0.9431	–	–	–	0.9407	–	–	–	0.5395	0.0911	0.0889	0.0895	0.9008	0.1221	0.1184	0.1157
22	0.9998	–	–	–	0.9996	–	–	–	0.8549	0.0884	0.0841	0.0864	0.9920	0.1723	0.1813	0.1792
23	0.9965	–	–	–	0.9968	–	–	–	0.7872	0.1026	0.0957	0.1007	0.9840	0.1038	0.1140	0.1091
24	0.9987	–	–	–	0.9984	–	–	–	0.8454	0.0841	0.0825	0.0838	0.9959	0.1726	0.1815	0.1793
25	1.0000	–	–	–	0.9999	–	–	–	0.9165	0.0873	0.0846	0.0857	0.9967	0.1795	0.1954	0.1862
26	0.9997	–	–	–	0.9999	–	–	–	0.8862	0.0911	0.0884	0.0967	0.9965	0.1575	0.1526	0.1428
27	0.9998	–	–	–	0.9999	–	–	–	0.8987	0.0871	0.0861	0.0901	0.9968	0.1785	0.1742	0.1676

Table 13: Power of the proposed and competing tests for alternatives LW(2), LW(0.5), W(2) and G(2) at $\alpha = 0.1$ when the null hypothesis is the Pareto distribution and $k = 3$.

CS	LW(2)				LW(0.5)				W(2)				G(2)			
	$T_\star$	D	W	A	$T_\star$	D	W	A	$T_\star$	D	W	A	$T_\star$	D	W	A
1	0.3374	0.0457	0.0452	0.0455	0.6471	0.1907	0.1920	0.1909	0.7839	–	–	–	0.7600	–	–	–
2	0.2290	0.1220	0.1210	0.1184	0.4217	0.0893	0.0873	0.0853	0.5395	–	–	–	0.5072	–	–	–
3	0.2459	0.1169	0.1198	0.1194	0.4839	0.0842	0.0869	0.0865	0.6124	–	–	–	0.5870	–	–	–
4	0.4825	0.0454	0.0426	0.0423	0.8118	0.2284	0.2345	0.2312	0.8985	–	–	–	0.8767	–	–	–
5	0.3574	0.1144	0.1090	0.1215	0.6568	0.0956	0.1031	0.1017	0.7730	–	–	–	0.7452	–	–	–
6	0.4655	0.0489	0.0487	0.0479	0.8140	0.2372	0.2433	0.2400	0.8564	–	–	–	0.8376	–	–	–
7	0.6027	0.0388	0.0367	0.0372	0.8942	0.2564	0.2722	0.2643	0.9453	–	–	–	0.9326	–	–	–
8	0.5148	0.0805	0.0859	0.1044	0.8312	0.1613	0.1616	0.1430	0.9067	–	–	–	0.8863	–	–	–
9	0.6286	0.0422	0.0394	0.0391	0.9123	0.2590	0.2753	0.2694	0.9419	–	–	–	0.9359	–	–	–
10	0.4779	0.0397	0.0355	0.0364	0.7678	0.2257	0.2291	0.2333	0.9183	–	–	–	0.9009	–	–	–
11	0.2655	0.1105	0.1182	0.1153	0.5027	0.0886	0.0903	0.0883	0.6291	–	–	–	0.6208	–	–	–
12	0.3356	0.1026	0.1032	0.1040	0.6355	0.0996	0.1072	0.1094	0.7500	–	–	–	0.7318	–	–	–
13	0.7723	0.0312	0.0286	0.0289	0.9618	0.3064	0.3247	0.3193	0.9912	–	–	–	0.9894	–	–	–
14	0.5826	0.1202	0.1299	0.1369	0.8691	0.0952	0.1074	0.1032	0.9417	–	–	–	0.9290	–	–	–
15	0.6841	0.0405	0.0392	0.0398	0.9482	0.2499	0.2616	0.2628	0.9737	–	–	–	0.9661	–	–	–
16	0.9047	0.0224	0.0192	0.0196	0.9927	0.3405	0.3739	0.3598	0.9994	–	–	–	0.9979	–	–	–
17	0.8210	0.1050	0.1106	0.1321	0.9775	0.1465	0.1640	0.1513	0.9963	–	–	–	0.9922	–	–	–
18	0.8582	0.0583	0.0702	0.0944	0.9864	0.2408	0.2364	0.2179	0.9972	–	–	–	0.9958	–	–	–
19	0.8138	0.0210	0.0184	0.0193	0.9633	0.3119	0.3235	0.3207	0.9957	–	–	–	0.9939	–	–	–
20	0.5566	0.1205	0.1294	0.1315	0.8433	0.0918	0.0960	0.0957	0.9309	–	–	–	0.9235	–	–	–
21	0.6811	0.0870	0.0829	0.0875	0.9323	0.1411	0.1327	0.1290	0.9686	–	–	–	0.9683	–	–	–
22	0.9684	0.0205	0.0148	0.0148	0.9993	0.4012	0.4368	0.4253	0.9998	–	–	–	0.9999	–	–	–
23	0.9161	0.1311	0.1231	0.1398	0.9946	0.1296	0.1498	0.1470	0.9990	–	–	–	0.9984	–	–	–
24	0.9550	0.0283	0.0262	0.0265	0.9990	0.3286	0.3514	0.3455	0.9998	–	–	–	0.9996	–	–	–
25	0.9880	0.0202	0.0130	0.0134	0.9999	0.4319	0.4846	0.4640	1.0000	–	–	–	1.0000	–	–	–
26	0.9758	0.0716	0.0854	0.1155	0.9998	0.2683	0.2554	0.2380	0.9998	–	–	–	1.0000	–	–	–
27	0.9789	0.0418	0.0545	0.0782	1.0000	0.3662	0.3432	0.3283	1.0000	–	–	–	1.0000	–	–	–