Journal of Data Science and Modeling, Vol. 3, No. 1, 143-174, December 2024

Research Manuscript

The unit linear exponential distribution: properties, quantile regression model and applications

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Recieved: 25/09/2025 Accepted: 01/03/2026

Abstract: The paper suggests a novel model, referred to as the unit linear exponential distribution. This model includes the unit exponential and unit Rayleigh distributions as special submodels. It has two unknown parameters, which are estimated using several estimation methods, including maximum likelihood, maximum product spacing, least squares, weighted least squares, Cramér-von Mises, and Anderson-Darling. The behavior of these estimators is assessed through a simulation study. In addition, a new quantile regression model based on the suggested distribution is introduced, with its parameters estimated via maximum likelihood. The modeling capability of the newly proposed distribution and the associated quantile regression model is proved using two well-known real-data applications.

Keywords: Unit distribution, linear exponential distribution, estimation methods, Monte Carlo simulations, Data analysis.

Mathematics Subject Classification (2010): 60E05, 62F10, 33B30.

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1. Introduction

In various fields such as reliability engineering, medical sciences, and economics, probability distributions are essential tools for modeling real data. The choice of an appropriate distribution depends on the nature of the data, including whether the data are bounded or unbounded, symmetric or skewed, discrete or continuous. If the data are limited to a closed interval, such as proportions, percentages, or standardized scores between 0 and 1, then a probability distribution with support on $(0, 1)$ is appropriate, and the beta and Kumaraswamy distributions are the most common choices; however, these distributions are not flexible enough for some applications, and therefore other alternatives defined on $(0, 1)$ to these distributions have been developed and are commonly used. These alternatives are typically constructed through appropriate transformations of unbounded distributions, like $X = exp(-Y)$, $X = 1/(1 + Y)$, $X = Y/(1 + Y)$, where Y is the unbounded random variable and X is a new unit random variable. For example: Menezes and *et al.* (2018) introduced the unit-logistic distribution using $X = Y/(1 + Y$ where Y has the logistic distribution. Mazucheli and *et al.* (2019) developed the unit-Gompertz distribution, using the transformation $X = exp(-Y)$, where Y has the Gompertz distribution. Mazucheli and *et al.* (2019) presented the unit-Lindley distribution, making use of the formula $X = Y/(1 + Y)$, where Y follows the Lindley distribution. Ragab and *et al.* (2024) presented the unit extended exponential distribution utilizing $X = exp(-Y)$, where Y has the extended exponential distribution. Maya and *et al.* (2024) defined the unit Muth model utilizing $X = exp(-Y)$, where Y has the Muth distribution. Sapkota and *et al.* (2025) suggested the unit Weibull distribution, utilizing the formula $X = 1/(1 + Y)$, where Y follows the Weibull distribution. Bashiru and *et al.* (2025) presented the unit Zeghdoudi distribution, using $X = Y/(1 + Y)$, where Y has the Zeghdoudi distribution.

The classical linear regression model assumes normally distributed errors over the entire real line, which makes it unsuitable for modeling response variables bounded in (0,1). Predictions from linear regression may fall outside this interval, and the assumption of constant variance (homoscedasticity) is often violated—particularly near the boundaries, where the variance of a proportion tends to shrink as values approach 0 or 1. To address these issues, Ferrari and Cribari-Neto (2004) introduced beta regression, a model based on the beta distribution, which naturally accommodates continuous outcomes strictly between 0 and 1 and allows for heteroscedasticity through a mean–precision parameterization.

Despite its popularity and flexibility, the beta regression model has notable limitations, such as inability to handle exact 0s and 1s (since the beta distribution has support on (0,1)), the assumption of a particular mean-variance relationship,

and limited flexibility in modeling complex data structures. To overcome these issues, various alternative models have been proposed, such as the Log-Lindley regression model Gomez-Deniz and *et al.* (2014), the Johnson SB regression model Lemont and Bazán (2016) and the unit-Lindley regression model Mazucheli and *et al.* (2019). When the response variable has outliers, quantile regression is often preferred over traditional (mean-based) regression modeling. This choice comes from the fact that the mean used in ordinary least squares regression is highly affected by extreme values and can be greatly distorted by biased or non-normal data (see Koenker and Bassett (1978). On the other hand, quantile regression, especially median regression, offers a more robust and informative option since it is less influenced by outliers. Through the re-parameterization of the probability distribution utilizing the quantile approach, numerous unit quantile regression models have been developed for response variables limited to the unit interval, like the Kumaraswamy regression model Mitnik and Baek (2013), the unit Burr-XII regression model Korkmaz and Chesneau (2021), the unit Chen regression model Korkmaz and *et al.* (2022), and the unit NET regression model Sağlam and Karakaya (2024). The current paper introduces a new quantile regression model as an alternative to these models.

The main motivation of our article is to present a new bounded distribution on (0,1) and a novel quantile regression model based on this distribution. To this end, we use the well-known linear exponential distribution, which is also called the linear failure rate distribution. If a random variable Y has a linear exponential distribution, then its cumulative distribution function (CDF) is given by

$$G(y) = 1 - e^{-ay-\frac{b}{2}y^2}, \tag{1.1}$$

where $y \geq 0$, $a \geq 0$ and $b \geq 0$ with $a + b > 0$. From (1.1) and Utilizing the transformation $X = e^{-Y}$, we get the unit linear exponential (ULE) distribution. After some simplifications, we obtain the CDF of the ULE distribution that is

$$F(x) = x^a e^{-\frac{b}{2}(\log x)^2}, \tag{1.2}$$

where $x \in (0,1)$, $a \geq 0$ and $b \geq 0$ with $a+b > 0$. The probability density function (PDF) of the ULE distribution is

$$f(x) = x^{a-1}(a - b\log x)e^{-\frac{b}{2}(\log x)^2}, \tag{1.3}$$

while its hazard rate function (HRF) is

$$h(x) = \frac{x^{a-1}(a - b\log x)}{e^{\frac{b}{2}(\log x)^2} - x^a}.$$

When $b = 0$, we get the unit exponential distribution, whereas when $a = 0$, the ULE distribution becomes the unit Rayleigh distribution with the parameter $\beta = b/2$ Bantan and *et al.* (2020). Hereafter, a random variable X is denoted by $X \sim ULE(a, b)$ if X has PDF (1.3).

To the best of our knowledge, this article presents the first introduction of the ULE distribution. The remainder of the paper is organized as follows. Section 2 outlines key properties of the ULE distribution. Section 3 introduces six distinct methods for estimating its unknown parameters. A comprehensive simulation study evaluating the performance of these estimation methods is provided in Section 4. Section 5 introduces the ULE quantile regression model. In Section 6, the flexibility and practical utility of both the ULE distribution and its regression model are demonstrated through applications to two real-world data sets. Finally, Section 7 offers concluding remarks.

2. Mathematical properties

2.1 Stochastic ordering property

The concept of stochastic ordering is a straightforward one that enables the comparison of distributions through their (CDFs). A distribution A is said to be stochastically larger than another distribution B if their CDFs, denoted $F_A(x)$ and $F_B(x)$, satisfy the condition: $F_A(x) \leq F_B(x)$ for all $x \in \mathbb{R}$. For more information, refer to Shaked and Shanthikumar (2007).

The following proposition allows us to conclude the stochastic ordering property satisfied by the ULE distribution.

Proposition 2.1. *For any $x \in (0, 1)$, the CDF of ULE distribution, $F_{a,b}(x)$, is decreasing and convex with respect to parameters a and b.*

Proof. We have

$$\frac{\partial}{\partial a} F_{a,b}(x) = (\log x) x^a e^{-\frac{a}{b}(\log x)^2}.$$

Since $x \in (0, 1)$, $\frac{\partial}{\partial a} F_{a,b}(x) < 0$, it means that $F_{a,b}(x)$ is decreasing with respect to a. On the other hand, we have

$$\frac{\partial^2}{\partial a^2} F_{a,b}(x) = (\log x)^2 x^a e^{-\frac{a}{b}(\log x)^2} > 0.$$

This indicates that $F_{a,b}(x)$ is convex with respect to a. Similarly, we have

$$\frac{\partial}{\partial b} F_{a,b}(x) = -\frac{1}{2} (\log x) x^a e^{-\frac{a}{b}(\log x)^2},$$

and

$$\frac{\partial^2}{\partial b^2}F_{a,b}(x) = \frac{1}{4}(\log x)^2 x^a e^{-\frac{a}{b}(\log x)^2}.$$

It is clear that $\frac{\partial}{\partial b}F_{a,b}(x) < 0$ and $\frac{\partial^2}{\partial b^2}F_{a,b}(x) > 0$. Hence, we conclude that $F_{a,b}(x)$ is decreasing and convex with respect to b. □

The Proposition 2.1 gives the following stochastic ordering property: If $a_2 \geq a_1$, then we have $F_{a_1,b}(x) \leq F_{a_2,b}(x)$. If $b_2 \geq b_1$, then we have $F_{a,b_1}(x) \leq F_{a,b_2}(x)$.

2.2 Shapes of the PDF and HRF

Here, the shapes of the PDF and HRF of the ULE distribution are discussed. The next proposition gives the shape and mode of the ULE distribution.

Proposition 2.2. *Let $X \sim ULE(a,b)$ with $a \geq 0$ and $b > 0$, then*

1. *The PDF satisfies,*

$$\lim_{x\to 0^+} f(x) = 0 \text{and} \lim_{x\to 1^-} f(x) = a.$$

2. *The PDF is unimodal, and its mode is given by*

$$x_{mode} = \exp\left(\frac{1}{2b}(2a - 1 - \sqrt{4b+1})\right).$$

Proof. 1. The First part follows directly from the expression of the PDF (1.3).

2. From (1.3), we have

$$\log f(x) = (a-1)\log x + \log(a - b\log x) - \frac{b}{2}(\log x)^2.$$

Differentiating with respect to x, we obtain

$$\frac{d\log f(x)}{dx} = \frac{b^2(\log x)^2 + b(1-2a)\log x + a^2 - a - b}{x(a - b\log x)}.$$

To obtain the mode of f, we solve the equation $\frac{d\log f(x)}{dx} = 0$. Taking $t = \log x$, we get the following quadratic equation

$$b^2t^2 + b(1-2a)t - (b - a^2 + a) = 0,$$

that has following two distinct roots

$$t = \frac{1}{2b}(2a - 1 \pm \sqrt{4b+1}),\ b > 0,$$

Since the domain of X is $(0,1)$, we require $t = \log x < 0$. The root with the minus sign yields a negative t (as $\sqrt{4b+1} > 1$ for $b > 0$), whereas the root with the plus sign yields $t > 0$ (as $4b + 1 > 1$, so $x > 1$, which lies outside the support. Therefore, the mode corresponds to the smaller root:

$$x_{\text{mode}} = \exp\left(\frac{1}{2b}\left(2a - 1 - \sqrt{4b+1}\right)\right).$$

This confirms unimodality, as the derivative of $\log f(x)$ changes sign from positive to negative at this point.

□

For the HRF of the ULE distribution: it is clear that for any $a \geq 0$ and $b > 0$, $\lim_{x\to 0^+} h(x) = 0$ and $\lim_{x\to 1^-} h(x) = +\infty$. On the other hand, we have

$$\frac{d\log h(x)}{dx} = \frac{b^2(\log x)^2 e^{\frac{b}{2}(\log x)^2} + \left(b(1-2a)e^{\frac{b}{2}(\log x)^2} - bx^a\right)\log x}{x\left(e^{\frac{b}{2}(\log x)^2} - x^a\right)(b\log x - a)} + \frac{(a^2 - a - b)e^{\frac{b}{2}(\log x)^2} + (b+a)x^a}{x\left(e^{\frac{b}{2}(\log x)^2} - x^a\right)(b\log x - a)}.$$

The analytical study of this equation is not straightforward. For some values of the parameters a and b, the PDF and HRF plots of the ULE distribution are shown in Figure 1. As shown in Figure 1 (left panel), the PDF of the ULE distribution is unimodal and can exhibit a decreasing or increasing. In Figure 1 (right panel), the HRF of the ULE distribution is observed to be flexible, showing various shapes including decreasing, increasing, bathub-shaped, or modified bathtub-shaped (unimodal followed by increasing). Hence, the ULE model can be used to fit various types of real data in different disciplines.

2.3 Quantile Function

Proposition 2.3. *The quantile function of the ULE distribution with parameters $a \geq 0$ and $b > 0$ is given by*

$$Q(u) = \exp\left(\frac{1}{b}(a - \sqrt{a^2 - 2b\log u})\right), \; u \in (0,1). \tag{2.4}$$

Proof. To find the quantile function of the ULE distribution, we solve the equation $u = F(x)$ according to x, where $F(x)$ is the CDF, (1.2), of the ULE distribution. After some algebraic operations, we obtain the desired result. □

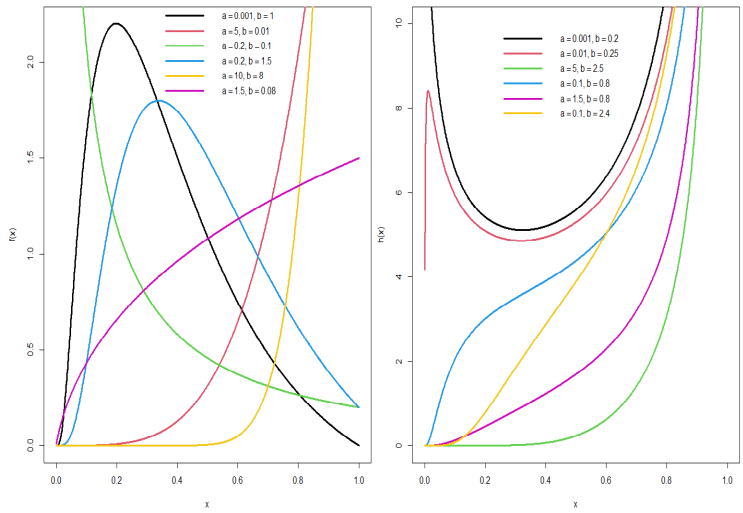

Figure 1: PDF (left) and HRF (right) of the ULE distribution for various parameter choices.

2.4 Incomplete Moments

If $X \sim ULE(a, b)$, we can obtain two expressions of the incomplete moments of X. The first is given in proposition 2.4 while the second is given in proposition 2.5.

Proposition 2.4. *If $X \sim ULE(a, b)$, then the r^{th} incomplete moment of X evaluated at $t \in (0, 1)$ is*

$$m_r(t) = t^{r+a} e^{\frac{-b(\log t)^2}{2}} - r\sqrt{\frac{\pi}{2b}} e^{\frac{(r+a)^2}{2b}} \operatorname{erfc}\left(\frac{a + r - b\log t}{\sqrt{2b}}\right),$$

where $\operatorname{erfc}(\eta)$ *is the complementary error function defined by* $\operatorname{erfc}(\eta) = \frac{2}{\sqrt{\pi}} \int_{\eta}^{+\infty} e^{-z^2}\, dz$, *with* $\eta \in \mathbb{R}$.

Proof. By definition

$$\begin{aligned} m_r(t) &= \int_0^t x^r f(x) dx \\ &= \int_0^t x^r x^{a-1} (a - b\log x) e^{-\frac{b}{2}(\log x)^2} dx. \end{aligned}$$

Using the substitution $y = -\log x$ (so that $x = e^{-y}$, $dx = -e^{-y}\, dy$), the limits change from $x = 0$ to $x = t$ into $y = \infty$ to $y = -\log t$, yielding

$$m_r(t) = \int_{-\log t}^{\infty} (a + by) \exp\left(-(r + a)y - \frac{b}{2}y^2\right) dy.$$

Complete the square in the exponent:

$$-(r+a)y - \frac{b}{2}y^2 = \frac{(r+a)^2}{2b} - \left(\frac{(r+a)+by}{\sqrt{2b}}\right)^2$$

Thus,

$$m_r(t) = \exp\left(\frac{(r+a)^2}{2b}\right)\int_{-\log t}^{+\infty}(a+by)\exp\left(-\left(\frac{(r+a)+by}{\sqrt{2b}}\right)^2\right)dy.$$

Now apply the substitution $z = \frac{r+a+by}{\sqrt{2b}}$, so that $y = \frac{\sqrt{2b}z-(r+a)}{b}$, and $dy = \frac{\sqrt{2b}}{b}\,dz$. The lower limit becomes $z_0 = \frac{r+a-b\log t}{\sqrt{2b}}$. Then:

$$m_r(t) = \exp\left(\frac{(r+a)^2}{2b}\right)\frac{\sqrt{2b}}{b}\int_{z_0}^{\infty}\left[a+b\left(\frac{\sqrt{2b}z-(r+a)}{b}\right)\right]e^{-z^2}\,dz.$$

Simplify the expression in bracket:

$$a+b\left(\frac{\sqrt{2b}z-(r+a)}{b}\right) = \sqrt{2b}z - r$$

Hence,

$$m_r(t) = \exp\left(\frac{(r+a)^2}{2b}\right)\frac{\sqrt{2b}}{b}\int_{z_0}^{\infty}\left[\sqrt{2b}z-r\right]e^{-z^2}\,dz.$$

Split the integral:

$$m_r(t) = \exp\left(\frac{(r+a)^2}{2b}\right)\left[2\int_{z_0}^{\infty}ze^{-z^2}\,dz - r\frac{\sqrt{2b}}{b}\int_{z_0}^{\infty}e^{-z^2}\,dz\right].$$

Note that:

$$\int_{z_0}^{\infty}ze^{-z^2}dz = \frac{1}{2}e^{-z_0^2}, \qquad \int_{z_0}^{\infty}e^{-z^2}dz = \frac{\sqrt{\pi}}{2}\operatorname{erfc}(z_0).$$

Therefore,

$$m_r(t) = \exp\left(\frac{(r+a)^2}{2b}\right)\left[e^{-z_0^2} - r\frac{\sqrt{2b}}{b}\sqrt{\frac{\pi}{2}}\operatorname{erfc}(z_0)\right].$$

But

$$e^{-z_0^2} = \exp\left(-\frac{(r+a-b\log t)^2}{2b}\right),$$

so

$$\exp\left(\frac{(r+a)^2}{2b}\right)\exp\left(-\frac{(r+a-b\log t)^2}{2b}\right) = \exp\left(-\frac{b(\log t)^2}{2}+(r+a)\log t\right) = t^{\,r+a}e^{-\frac{b(\log t)^2}{2}}.$$

Thus,

$$m_r(t) = t^{r+a} e^{\frac{-b(\log t)^2}{2}} - r\sqrt{\frac{\pi}{2b}} e^{\frac{(r+a)^2}{2b}} \operatorname{erfc}\left(\frac{a+r-b\log t}{\sqrt{2b}}\right),$$

Which completes the proof. □

From Proposition 2.4, we have $m_0(t) = F(t) = t^a e^{-\frac{(b(\log t)^2)}{2}}$ as expected, taking $r = 0$, with $t \in (0,1)$. The following Proposition 2.5 gives an alternate expression of the r^{th} incomplete moments.

Proposition 2.5. *If $X \sim ULE(a,b)$, then the r^{th} incomplete moment of X at $t \in (0,1)$, for $a + r > 0$ is*

$$m_r(t) = \sum_{k=0}^{+\infty} \frac{(-1)^k b^k}{2^k k!} \left(\frac{a\Gamma(2k+1, -(a+r)\log t)}{(r+a)^{2k+1}} + \frac{b\Gamma(2k+2, -(a+r)\log t)}{(r+a)^{2k+2}}\right),$$

where $\Gamma(\theta, z)$ is the incomplete upper gamma function defined by $\Gamma(\theta, z) = \int_z^{+\infty} t^{\theta-1} e^{-t} dt$ for $\theta, z > 0$.

Proof. Starting again from

$$m_r(t) = \int_0^t x^r x^{a-1} (a - b\log x) e^{-(b/2)(\log x)^2} dx.$$

Let $y = -\log x$, so $x = e^{-y}$, $dx = -e^{-y}\, dy$, and $x \in (0,t) \implies y \in (-\log t, \infty)$. Then

$$m_r(t) = \int_{-\log(t)}^{+\infty} (a + by) e^{-(r+a)y} e^{-(b/2)y^2} dy.$$

Now expand of $e^{-\frac{b}{2}y^2}$ as a power series:

$$e^{-\frac{b}{2}y^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{b}{2}\right)^k y^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k b^k}{2^k k!} y^{2k},$$

which is valid for all $y \in \mathbb{R}$. Interchanging sum and integral (justified by absolute convergence for $t \in (0,1)$), we obtain

$$m_r(t) = \sum_{k=0}^{\infty} \frac{(-1)^k b^k}{2^k k!} \int_{-\log t}^{\infty} (a + by) y^{2k} e^{-(r+a)y}\, dy.$$

Now make the change of variable $z = (r+a)y$, so that $y = \frac{z}{r+a}$ and $dy = \frac{dz}{r+a}$.

Then:

$$\begin{aligned}\int_{-\log t}^{\infty} y^{2k}(a+by)e^{-(r+a)y}dy &= \frac{a}{(r+a)^{2k+1}}\int_{-(r+a)\log t}^{\infty} z^{2k}e^{-z}dz \\ &+ \frac{b}{(r+a)^{2k+2}}\int_{-(r+a)\log t}^{\infty} z^{2k+1}e^{-z}dz \\ &= \frac{a\Gamma(2k+1,-(a+r)\log t)}{(r+a)^{2k+1}} \\ &+ \frac{b\Gamma(2k+2,-(a+r)\log t)}{(r+a)^{2k+2}}.\end{aligned}$$

Substituting back yields the desired result. □

2.5 Moments

Since the r^{th} moments of X about the origin verify

$$\mu_r = \lim_{t\to 1} m_r(t), \tag{2.5}$$

then from Propositions 2.4 and 2.5, we have the following propositions.

Proposition 2.6. *If $X \sim ULE(a,b)$, then the r^{th} the moment of X about the origin, for $a+r>0$ is*

$$\mu_r = 1 - r\sqrt{\frac{\pi}{2b}}e^{\frac{(r+a)^2}{2b}}\operatorname{erfc}\left(\frac{a+r}{\sqrt{2b}}\right),$$

where erfc(.) *is given in Proposition 2.4.*

Proposition 2.7. *If $X \sim ULE(a,b)$, then*

$$\mu_r = \sum_{k=0}^{+\infty}\frac{(-1)^k b^k}{2^k k!}\left(a\frac{(2k)!}{(r+a)^{2k+1}} + b\frac{(2k+1)!}{(r+a)^{2k+2}}\right).$$

2.6 Probability Weighted Moment

The $(r,s)^{th}$ probability weighted moment, denoted by $m_{r,s}$, where r and s are non-negative integers, may be considered as extensions of the moments. The closed forms of $m_{r,s}$ for the ULE distribution are given below.

Proposition 2.8. *If $X \sim$ ULE(a,b), then the $(r,s)^{\text{th}}$ probability weighted moment of X is given by*

$$m_{r,s} = \frac{1}{s+1}\left[1 - r\sqrt{\frac{\pi}{2b(s+1)}}\exp\left(\frac{\big(a(s+1)+r\big)^2}{2b(s+1)}\right)\operatorname{erfc}\left(\frac{a(s+1)+r}{\sqrt{2b(s+1)}}\right)\right],$$

where erfc$(\cdot)$ *denotes the complementary error function, as defined in Proposition 2.4.*

Proof. By definition

$$m_{r,s} = \int_0^1 x^r f(x) F^s(x) dx,$$

where

$$\begin{aligned} f(x)F^s(x) &= x^{a(s+1)-1}(a - b\log x)e^{-\frac{b(s+1)(\log x)^2}{2}} \\ &= \frac{1}{s+1}x^{a(s+1)-1}((s+1)a - b(s+1)\log x)e^{-\frac{b(s+1)(\log x)^2}{2}} \\ &= \frac{1}{s+1}f_Z(x), \end{aligned}$$

It is clear that f_Z is the PDF of $Z \sim ULE\left(((s+1)a, (s+1)b)\right)$. So

$$\begin{aligned} m_{r,s} &= \frac{1}{s+1}\int_0^1 x^r f_Z(x)\, dx \\ &= \frac{1}{s+1}E(Z^r). \end{aligned}$$

From Proposition 2.6, , for $a(s+1) + r > 0$, we have

$$E(Z^r) = 1 - r\sqrt{\frac{\pi}{2b(s+1)}}e^{\frac{(a(s+1)+r)^2}{2b(s+1)}}\operatorname{erfc}\left(\frac{a(s+1)+r}{\sqrt{2b(s+1)}}\right)$$

Therefore, we can conclude the desired result. □

Proposition 2.9. *If $X \sim ULE(a, b)$, then the $(r, s)^{th}$ probability weighted moments of X, for $a + r > 0$ is*

$$m_{r,s} = \sum_{k=0}^{+\infty} \frac{(-1)^k b^k}{2^k k!(s+1)}\left(a\frac{(2k)!}{(r+a)^{2k+1}} + b\frac{(2k+1)!}{(r+a)^{2k+2}}\right).$$

Proof. Similarly, from the proof of Proposition 2.8, we have

$$m_{r,s} = \frac{1}{s+1}E(Z^r).$$

Using Proposition 2.7, we can conclude the desired result. □

2.7 Order statistics

If $X_1, \ldots, X_n$ is a random sample and $X_{1,n}, \ldots, X_{n,n}$ are the order statistics from this sample, then the PDF of $X_{\ell,n}$ is given by

$$f_\ell(x) = \frac{n!}{(n-\ell)!(\ell-1)!}f(x)(F(x))^{\ell-1}(1 - F(x))^{n-\ell}.$$

If $X \sim \text{ULE}(a, b)$, then the PDF of $X_{\ell,n}$ is

$$f_\ell(x) = \frac{n!(a - b\log x)}{(n-\ell)!(\ell-1)!} x^{a\ell-1} e^{-\frac{b\ell}{2}(\log x)^2} (1 - x^a e^{-\frac{b}{2}(\log x)^2})^{n-\ell}.$$

The r^{th} moment of $X_{\ell,n}$ can be obtained using the probability weighted moments of X. Since $0 < F(x) < 1$, using the binomial formula of $[1 - F(x)]^{n-\ell}$, we can obtain

$$f_{\ell,n}(x) = \frac{n!}{(n-\ell)!(\ell-1)!} \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} (-1)^j f(x)[F(x)]^{\ell+j-1}.$$

The r^{th} moment of $X_{\ell,n}$ is

$$\mu_{r,\ell}(x) = \frac{n!}{(n-\ell)!(\ell-1)!} \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} (-1)^j \int_0^1 x^r f(x)[F(x)]^{\ell+j-1}$$

$$= \frac{n!}{(n-\ell)!(\ell-1)!} \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} (-1)^j m_{r,\ell+j-1},$$

where, from Proposition 2.8,

$$m_{r,\ell+j-1} = \frac{1}{s+1} \left(1 - r\sqrt{\frac{\pi}{2b(\ell+j)}} \exp\left(\frac{(a(\ell+j)+r)^2}{2b(\ell+j)}\right) \operatorname{erfc}\left(\frac{a(\ell+j)+r}{\sqrt{2b(\ell+j)}}\right)\right).$$

2.8 Stochastic orderings

If F_X and F_Y are the CDFs and f_X and f_Y are the PDFs of the variables X and Y, then X is stochastically smaller or equal to Y denoted by $X \leq_{st} Y$, if $F_X(x) \leq F_Y(x)$ for all x. Similarly, we have

- the likelihood ratio order, $X \leq_{lr} Y$, if $\frac{f(x)}{f_Y(x)}$ is decreasing in x,
- the reversed hazard rate order $X \leq_{rhr} Y$, if $F_X(x)/F_Y(x)$ is decreasing in x.
- the hazard rate order $X \leq_{\text{hr}} Y$, if $\frac{1-F_X(x)}{1-F_Y(x)}$ is decreasing in x.

From Shaked and Shanthikumar (2007) and Benkhelifa (2022), we have

$$X \leq_{rhr} Y \Leftarrow X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \tag{2.6}$$

Proposition 2.10. *Let $X \sim ULE(a_1, b)$ and $Y \sim ULE(a_2, b)$. If $a_2 > a_1$, then:*

$$X \leq_{lr} Y, \quad X \leq_{rhr} Y, \quad X \leq_{hr} Y \quad \text{and} \quad X \leq_{st} Y.$$

Proof. The logarithmic likelihood ratio of X and Y is given by

$$\log\left(\frac{f_X(x)}{f_Y(x)}\right) = \log\left(\frac{x^{a_1-1}(a_1 - b\log x)}{x^{a_2-1}(a_2 - b\log x)}\right).$$

It is clear that

$$\frac{d}{dx}\log\left(\frac{f_X(x)}{f_Y(x)}\right) = \frac{(a_1-a_2)(b^2\ln^2 x - (a_1+a_2)b\ln x + b + a_1a_2)}{x(a_1 - b\ln x)(a_2 - b\ln x)}.$$

Since $x \in (0,1)$, we have $\frac{d}{dx}\log\left(\frac{f_X(x)}{f_Y(x)}\right) < 0$ only if $a_2 > a_1$. Therefore, $\frac{f_X(x)}{f_Y(x)}$ decreases in x, i.e., $X \leq_{lr} Y$. The remaining statements can be obtained from the implications 2.6. □

2.9 Stress strength reliability

Proposition 2.11. *Let $X_1 \sim ULE(a_1, b_1)$ and $X_2 \sim ULE(a_2, b_2)$, where X_1 and X_2 are independent random variables, then*

$$R = \frac{\sqrt{2\pi}(a_1b_2 - a_2b_1)}{2(b_1+b_2)^{(3/2)}}\exp\left(\frac{(a_1+a_2)^2}{2(b_1+b_2)}\right) erfc\left(\frac{a_1+a_2}{\sqrt{2(b_1+b_2)}}\right) + \frac{b_1}{b_1+b_2}$$

Proof. We have

$$\begin{aligned} R &= \int_0^1 f_{X_1}(x, a_1, b_1)F_{X_2}(x, a_2, b_2)\,dx \\ &= \int_0^1 x^{a_1+a_2-1}(a_1 - b_1\log x)e^{-\left(\frac{b_1+b_2}{2}\right)(\log x)^2}\,dx \end{aligned}$$

Substituting the variable $y = -\log x$ into the last integral, we get

$$\begin{aligned} R = a_1\int_0^{+\infty}\exp\left(-(a_1+a_2)y - \frac{b_1+b_2}{2}y^2\right)dy \\ + b_1\int_0^{+\infty} y\exp\left(-(a_1+a_2)y - \frac{b_1+b_2}{2}y^2\right)dy \end{aligned}$$

and then, after some algebraic operations, we get

$$\begin{aligned} R = a_1\exp\left(\frac{(a_1+a_2)^2}{2(b_1+b_2)}\right)\int_0^{+\infty}\exp\left(-\left(\frac{(a_1+a_2)+(b_1+b_2)y}{\sqrt{2(b_1+b_2)}}\right)^2\right)dy \\ + b_1\exp\left(\frac{(a_1+a_2)^2}{2(b_1+b_2)}\right)\int_0^{+\infty} y\exp\left(-\left(\frac{(a_1+a_2)+(b_1+b_2)y}{\sqrt{2(b_1+b_2)}}\right)^2\right)dy. \end{aligned}$$

Applying the change of variable $z = \frac{(a_1+a_2)+(b_1+b_2)y}{\sqrt{2(b_1+b_2)}}$, and after some algebraic operations, we obtain

$$R = a_1 \frac{\sqrt{2(b_1+b_2)}}{b_1+b_2} \exp\left(\frac{(a_1+a_2)^2}{2(b_1+b_2)}\right) \int_{\frac{a_1+a_2}{\sqrt{2(b_1+b_2)}}}^{+\infty} e^{-z^2} dz$$
$$+ b_1 \frac{\sqrt{2(b_1+b_2)}}{b_1+b_2} \exp\left(\frac{(a_1+a_2)^2}{2(b_1+b_2)}\right) \int_{\frac{a_1+a_2}{\sqrt{2(b_1+b_2)}}}^{+\infty} \left(\frac{z\sqrt{2(b_1+b_2)} - (a_1+a_2)}{b_1+b_2}\right) e^{-z^2} dz.$$

On other hand, we have

$$\int_{\frac{a_1+a_2}{\sqrt{2(b_1+b_2)}}}^{\infty} \left(\frac{z\sqrt{2(b_1+b_2)} - (a_1+a_2)}{b_1+b_2}\right) e^{-z^2} dz = -\frac{a_1+a_2}{b_1+b_2} \int_{\frac{a_1+a_2}{\sqrt{2(b_1+b_2)}}}^{\infty} e^{-z^2} dz$$
$$+ \frac{\sqrt{2(b_1+b_2)}}{2(b_1+b_2)} \exp\left(-\frac{(a_1+a_2)^2}{2(b_1+b_2)}\right).$$

Therefore, by definition of erfc(.), we obtain the desired result. □

Proposition 2.12. *Let $X_1 \sim ULE(a_1, b_1)$ and $X_2 \sim ULE(a_2, b_2)$, where X_1 and X_2 are independent random variables, then*

$$R = \sum_{k=0}^{+\infty} \frac{(-1)^k (b_1+b_2)^k}{2^k k!} \left(a_1 \frac{(2k)!}{(a_1+a_2)^{2k+1}} + b_1 \frac{(2k+1)!}{(a_1+a_2)^{2k+2}}\right).$$

Proof. From the above proof, we have

$$R = a_1 \int_0^{+\infty} e^{-(a_1+a_2)y} e^{-\frac{b_1+b_2}{2}y^2} dy + b_1 \int_0^{+\infty} y e^{-(a_1+a_2)y} e^{-\frac{b_1+b_2}{2}y^2} dy.$$

Applying the Taylor series expansion of $e^{-\frac{b_1+b_2}{2}y^2}$, we get

$$R = \sum_{k=0}^{+\infty} \frac{(-1)^k (b_1+b_2)^k}{2^k k!} \left(a_1 \int_0^{+\infty} y^{2k} e^{-(a_1+a_2)y} dy + b_1 \int_0^{+\infty} y^{2k+1} e^{-(a_1+a_2)y} dy\right).$$

Applying the change of variable $z = (a_1+a_2)y$, we get

$$R = \sum_{k=0}^{+\infty} \frac{(-1)^k (b_1+b_2)^k}{2^k k!} \left(\frac{a_1}{(a_1+a_2)^{2k+1}} \int_0^{+\infty} z^{2k} e^{-z} dz + \frac{b_1}{(a_1+a_2)^{2k+2}} \int_0^{+\infty} z^{2k+1} e^{-z} dz\right)$$
$$= \sum_{k=0}^{+\infty} \frac{(-1)^k (b_1+b_2)^k}{2^k k!} \left(a_1 \frac{(2k)!}{(a_1+a_2)^{2k+1}} + b_1 \frac{(2k+1)!}{(a_1+a_2)^{2k+2}}\right).$$

□

2.10 Tsallis and Rényi entropies

Proposition 2.13. *If $X \sim \mathrm{ULE}(a,b)$, then for $a>0, b>0, s>0$ and $s\neq 1$, the Tsallis entropy of X is*

$$T_s = \frac{1}{s-1}\left[1 - \sum_{k=0}^{\infty}\sum_{l=0}^{2k} \frac{(-1)^{k+l}\binom{2k}{l} a^l (sb)^k b^{s-l}}{2^k k!(sa-s+1)^{2k-l+s+1}} \times e^{\frac{(sa-s+1)a}{b}}\,\Gamma\!\left(2k-l+s+1, \frac{(sa-s+1)a}{b}\right)\right].$$

where $\Gamma(\alpha,z)=\int_z^\infty t^{\alpha-1}e^{-t}\,dt$ is the upper incomplete gamma function.

Proof. The Tsallis entropy of X is defined as

$$T_s = \frac{1}{s-1}\left(1 - \int_0^1 f^s(x)\,dx\right), \quad s>0,\ s\neq 1,$$

where the density of the ULE distribution is such that

$$f^s(x) = x^{s(a-1)}(a-b\log x)^s \exp\!\left(-\frac{sb}{2}(\log x)^2\right).$$

Apply the change of variable $y=-\log x$ (so that $x=e^{-y}$, $dx=-e^{-y}dy$), which transforms the integral into

$$\int_0^1 f^s(x)\,dx = \int_0^\infty (a+by)^s \exp\big(-(s(a-1)+1)y\big)\exp\!\left(-\frac{sb}{2}y^2\right)dy.$$

Use Taylor series:

$$\exp\!\left(-\frac{sb}{2}y^2\right) = \sum_{k=0}^{\infty}\frac{(-1)^k(sb)^k}{2^k k!}y^{2k}.$$

Thus,

$$\int_0^\infty (a+by)^s e^{-(s(a-1)+1)y} e^{-\frac{sb}{2}y^2}\,dy = \sum_{k=0}^{\infty}\frac{(-1)^k(sb)^k}{2^k k!}\int_0^\infty (a+by)^s y^{2k} e^{-cy}\,dy,$$

where $c=s(a-1)+1=sa-s+1$. Substituting $t=a+by \quad\Rightarrow\quad y=\frac{t-a}{b}, \quad dy=\frac{dt}{b}$, as $y=0$, $t=a$; and as $y\to\infty$, $t\to\infty$. We have:

$$\int_0^\infty (a+by)^s y^{2k} e^{-cy}\,dy = \frac{e^{ca/b}}{b^{2k+1}}\int_a^\infty t^s (t-a)^{2k} e^{-(c/b)t}\,dt.$$

Now expand $(t-a)^{2k}$ using the binomial theorem:

$$(t-a)^{2k} = \sum_{l=0}^{2k}\binom{2k}{l}(-1)^l a^l t^{2k-l}.$$

Therefore:

$$\int_a^\infty t^s(t-a)^{2k}e^{-\frac{c}{b}t}\,dt = \sum_{l=0}^{2k}\binom{2k}{l}(-1)^l a^l \int_a^\infty t^{s+2k-l}e^{-\frac{c}{b}t}$$
$$= \sum_{l=0}^{2k}\binom{2k}{l}(-1)^l a^l \left(\frac{b}{c}\right)^{-(s+2k-l+1)} \Gamma\Big(s+2k-l+1, \frac{ca}{b}\Big).$$

We recall

$$\int_0^\infty (a+by)^s y^{2k}e^{-cy}\,dy = \frac{e^{ca/b}}{b^{2k+1}}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^l a^l \left(\frac{c}{b}\right)^{-(s+2k-l+1)} \Gamma\Big(s+2k-l+1, \frac{ca}{b}\Big).$$

So, after the simplification, one get

$$\int_0^1 f^s(x)\,dx = e^{\frac{ca}{b}}\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\frac{(-1)^{k+l}\binom{2k}{l}a^l(sb)^k b^{\,s-l}\,\Gamma\big(s+2k-l+1,\frac{ca}{b}\big)}{2^k\,k!\,c^{\,s+2k-l+1}}.$$

Finally, recall $c = sa - s + 1$, we obtain the expression of T_s.

□

The Rényi entropy of order $\tau > 0$, $\tau \neq 1$, is defined as

$$I_R(\tau) = \frac{1}{1-\tau}\log\left(\int_0^1 f^\tau(x)\,dx\right).$$

From the above proof, (replacing s with τ), we obtain

$$I_R(\tau) = \frac{1}{1-\tau}\log\left\{\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\frac{(-1)^{k+l}\binom{2k}{l}a^l(\tau b)^k b^{\tau-l}}{2^k k!\,(\tau a-\tau+1)^{\,2k-l+\tau+1}}\right.$$
$$\left.\times\, e^{\frac{(\tau a-\tau+1)a}{b}}\,\Gamma\left(2k-l+\tau+1,\ \frac{(\tau a-\tau+1)a}{b}\right)\right\}.$$

In the limit $\tau \to 1$, the Rényi entropy converges to the Shannon entropy:

$$\lim_{\tau\to 1} I_R(\tau) = -\int_0^1 f(x)\log f(x)\,dx.$$

3. Methods of parameter estimation

We estimate the parameters a and b of the ULE distribution using six methods.

3.1 Maximum likelihood estimation

The log-likelihood function for n observed values $x_1, \ldots, x_n$ from $X \sim \text{ULE}(a, b)$ is

$$L(a,b) = (a-1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log\big(a - b\log(x_i)\big) - \frac{b}{2}\sum_{i=1}^{n}\big(\log(x_i)\big)^2.$$

To find the maximum likelihood estimators (MLEs) $\hat{a}$ and $\hat{b}$ of a and b, respectively, we solve the nonlinear system:

$$\frac{\partial L(a,b)}{\partial a} = 0, \quad \frac{\partial L(a,b)}{\partial b} = 0,$$

where

$$\frac{\partial L(a,b)}{\partial a} = \sum_{i=1}^{n}\frac{1}{a - b\log(x_i)} + \sum_{i=1}^{n}\log(x_i),$$

and

$$\frac{\partial L(a,b)}{\partial b} = \sum_{i=1}^{n}\frac{-\log(x_i)}{a - b\log(x_i)} - \frac{1}{2}\sum_{i=1}^{n}\big(\log(x_i)\big)^2.$$

Since these equations do not have a closed-form solution, it is feasible to use the statistical software to perform numerical evaluation through iterative techniques, like the Newton-Raphson algorithm.

It is well known that the MLEs $\hat{a}$ and $\hat{b}$ are jointly asymptotically normal with mean (a, b) and asymptotic covariance matrix $\mathbf{J}_n^{-1}(a, b)$, where $\mathbf{J}_n(a, b)$ is the observed Fisher information matrix:

$$\mathbf{J}_n(a,b) = -\begin{pmatrix} \sum\limits_{i=1}^{n}\dfrac{-1}{\big[a - b\log(x_i)\big]^2} & \sum\limits_{i=1}^{n}\dfrac{\log(x_i)}{\big[a - b\log(x_i)\big]^2} \\ \sum\limits_{i=1}^{n}\dfrac{\log(x_i)}{\big[a - b\log(x_i)\big]^2} & \sum\limits_{i=1}^{n}\dfrac{-\big[\log(x_i)\big]^2}{\big[a - b\log(x_i)\big]^2} \end{pmatrix}.$$

Using this asymptotic result, approximate $(1-\zeta)\times 100\%$ confidence intervals for a and b are given by

$$\hat{a} \pm z_{\zeta/2}\sqrt{\widehat{V}(\hat{a})}, \quad \hat{b} \pm z_{\zeta/2}\sqrt{\widehat{V}(\hat{b})},$$

where $\widehat{V}(\hat{a})$ and $\widehat{V}(\hat{b})$ are the diagonal elements of $\mathbf{J}_n^{-1}(\hat{a}, \hat{b})$, and $z_{\zeta/2}$ denotes the $(1-\zeta/2)$-quantile of the standard normal distribution.

3.2 Maximum Product Spacing Estimation

Let $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$ denote the ordered observations from a random sample $X \sim \text{ULE}(a, b)$. Below, we describe several estimation methods for the parameters

a and b. According to Cheng and Amin (1979), the maximum product spacing estimators (MPSEs), denoted a_{MPS} and b_{MPS}, are obtained by solving the system:

$$\frac{\partial \mathrm{MPS}(a,b)}{\partial a} = 0, \quad \frac{\partial \mathrm{MPS}(a,b)}{\partial b} = 0,$$

where

$$\frac{\partial \mathrm{MPS}(a,b)}{\partial a} = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{F'_a(x_{i:n};a,b) - F'_a(x_{i-1:n};a,b)}{F(x_{i:n};a,b) - F(x_{i-1:n};a,b)} = 0,$$
$$\frac{\partial \mathrm{MPS}(a,b)}{\partial b} = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{F'_b(x_{i:n};a,b) - F'_b(x_{i-1:n};a,b)}{F(x_{i:n};a,b) - F(x_{i-1:n};a,b)} = 0,$$

with the partial derivatives of the CDF given by

$$F'_a(x;a,b) = (\log x)\, x^a e^{-\frac{b}{2}(\log x)^2}, \tag{3.7}$$
$$F'_b(x;a,b) = -\frac{1}{2} x^a (\log x)^2 e^{-\frac{b}{2}(\log x)^2}. \tag{3.8}$$

3.3 Least squares estimation

Let a_{LSE} and b_{LSE} denote the least squares estimates (LSEs), which minimize

$$\mathrm{LSE}(a,b) = \sum_{i=1}^{n} \left(F(x_{i:n};a,b) - \frac{i}{n+1} \right)^2.$$

The estimators satisfy

$$\frac{\partial \mathrm{LSE}(a,b)}{\partial a} = 2\sum_{i=1}^{n} F'_a(x_{i:n};a,b) \left(F(x_{i:n};a,b) - \frac{i}{n+1} \right) = 0,$$
$$\frac{\partial \mathrm{LSE}(a,b)}{\partial b} = 2\sum_{i=1}^{n} F'_b(x_{i:n};a,b) \left(F(x_{i:n};a,b) - \frac{i}{n+1} \right) = 0,$$

where F'_a and F'_b are given in (3.7) and (3.8), respectively.

3.4 Weighted least squares estimation

The weighted least squares estimates (WLSEs), denoted a_{WLSE} and b_{WLSE}, minimize a weighted version of the least squares criterion. They are solutions to:

$$\frac{\partial \mathrm{WLSE}(a,b)}{\partial a} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} F'_a(x_{i:n};a,b) \left(F(x_{i:n};a,b) - \frac{i}{n+1} \right) = 0,$$
$$\frac{\partial \mathrm{WLSE}(a,b)}{\partial b} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} F'_b(x_{i:n};a,b) \left(F(x_{i:n};a,b) - \frac{i}{n+1} \right) = 0,$$

with F'_a and F'_b as in (3.7) and (3.8).

3.5 Cramér–von Mises estimation

The Cramér–von Mises estimates (CVMEs), a_{CVME} and b_{CVME}, minimize

$$\text{CVME}(a,b) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_{i:n};a,b) - \frac{2i-1}{2n}\right)^2.$$

The corresponding estimating equations are:

$$\frac{\partial \text{CVME}(a,b)}{\partial a} = 2\sum_{i=1}^{n} F'_a(x_{i:n};a,b)\left(F(x_{i:n};a,b) - \frac{2i-1}{2n}\right) = 0,$$
$$\frac{\partial \text{CVME}(a,b)}{\partial b} = 2\sum_{i=1}^{n} F'_b(x_{i:n};a,b)\left(F(x_{i:n};a,b) - \frac{2i-1}{2n}\right) = 0,$$

with F'_a and F'_b given by (3.7) and (3.8).

3.6 Anderson–Darling estimation

The Anderson–Darling estimates (ADEs), a_{ADE} and b_{ADE}, minimize

$$\text{ADE}(a,b) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{i:n};a,b) + \log\big(1 - F(x_{i:n};a,b)\big)\right].$$

The score equations are:

$$\frac{\partial \text{ADE}(a,b)}{\partial a} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\frac{F'_a(x_{i:n};a,b)}{F(x_{i:n};a,b)} - \frac{F'_a(x_{i:n};a,b)}{1-F(x_{i:n};a,b)}\right) = 0,$$
$$\frac{\partial \text{ADE}(a,b)}{\partial b} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\frac{F'_b(x_{i:n};a,b)}{F(x_{i:n};a,b)} - \frac{F'_b(x_{i:n};a,b)}{1-F(x_{i:n};a,b)}\right) = 0,$$

where again F'_a and F'_b are defined in (3.7) and (3.8).

4. Simulation study

Here, to assess the behavior of the previously mentioned estimators (see Section 3) of the parameters a and b of the ULE distribution. For each simulation, we generate 1000 independent samples of size $n = 20, \ldots, 1000$, drawn from the ULE distribution using the true values of a and b. We set $a = 1.5$ and $b = 3.5$ in the first simulation while $a = 5.5$ and $b = 2.5$ in the second. Each random sample is produced by employing the empirical quantile function (2.4). The examination is based on the bias and mean square error (MSE), which are

$$\text{Bias}_\lambda(n) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\lambda}_i - \lambda),$$
$$\text{MSE}_\lambda(n) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\lambda}_i - \lambda)^2,$$

where $\lambda \in \{a, b\}$. Tables 1 and 2 present the results of our simulation study. As expected, both the bias and MSE decrease to zero as n increases for all proposed methods, confirming their consistency. Among the methods considered, the MLE demonstrates superior performance, particularly for larger sample sizes.

Table 1: Simulation results of the URL distribution for $a = 1.5$ and $b = 3.5$.

n	Metric	Param	MLE	MPS	LS	WLS	CVM	AD
20	Bias	a	0.3263	0.2202	0.2235	0.4397	0.7842	0.5674
		b	0.2655	0.2333	0.3090	0.3897	0.4304	0.2521
	MSE	a	0.1690	0.1005	0.1324	0.1688	0.1811	0.1102
		b	0.2850	0.2322	0.3510	0.3899	0.3887	0.3669
50	Bias	a	0.2974	0.2019	0.2002	0.4125	0.7666	0.5501
		b	0.2341	0.2099	0.2987	0.3613	0.4157	0.2329
	MSE	a	0.1557	0.0985	0.1176	0.1589	0.1721	0.1009
		b	0.2509	0.2200	0.3421	0.3602	0.3701	0.3522
200	Bias	a	0.1158	0.1599	0.1310	0.2125	0.3897	0.3442
		b	0.1542	0.1536	0.1974	0.3308	0.3614	0.1718
	MSE	a	0.0458	0.0579	0.0891	0.1021	0.1303	0.0812
		b	0.0510	0.1338	0.1422	0.1263	0.1641	0.1498
400	Bias	a	0.0963	0.1020	0.0785	0.1617	0.2654	0.2001
		b	0.0902	0.0986	0.1002	0.2011	0.2202	0.0974
	MSE	a	0.0232	0.0322	0.0791	0.0944	0.1017	0.0621
		b	0.0102	0.1012	0.1120	0.1080	0.1433	0.1215
600	Bias	a	0.0560	0.0757	0.0487	0.1210	0.1004	0.1324
		b	0.0710	0.0801	0.0902	0.1124	0.1511	0.0745
	MSE	a	0.0110	0.0191	0.0511	0.0614	0.0791	0.0354
		b	0.0098	0.0988	0.0974	0.0971	0.1079	0.1001
1000	Bias	a	0.0016	0.0021	0.0034	0.0081	0.0100	0.0167
		b	0.0097	0.0033	0.0098	0.0199	0.0210	0.0082
	MSE	a	0.0016	0.0091	0.0300	0.0169	0.0327	0.0123
		b	0.0022	0.0343	0.0574	0.0700	0.0921	0.0889

Table 2: Simulation results of the URL distribution for $a = 5.5$ and $b = 2.5$.

n	Metric	Param	MLE	MPS	LS	WLS	CVM	AD
20	Bias	a	0.2328	0.2366	1.0998	1.1632	1.1775	1.1587
		b	0.1217	0.1225	0.1522	0.1325	0.2954	0.1459
	MSE	a	2.1028	2.8540	3.0054	3.3236	3.4521	3.2948
		b	1.9984	2.2564	2.9577	3.1546	3.4073	3.2201
50	Bias	a	0.2116	0.2100	1.0024	1.1452	1.1642	1.1310
		b	0.1103	0.2221	0.1012	0.1136	0.2621	0.1235
	MSE	a	2.0090	2.6231	2.9627	3.2220	3.2931	3.1965
		b	1.7851	2.2034	2.8560	3.1010	3.3028	3.1774
200	Bias	a	0.1698	0.1747	0.6022	1.0541	1.1210	1.1017
		b	0.0874	0.1852	0.0984	0.8532	0.2124	0.0998
	MSE	a	1.2505	2.0657	2.2549	2.8402	3.0011	2.9980
		b	1.1594	1.9887	2.3001	2.2298	3.0009	2.9953
400	Bias	a	0.1120	0.1311	0.4235	1.0001	1.0154	0.9871
		b	0.0526	0.1500	0.0654	0.4556	0.1534	0.0785
	MSE	a	0.9111	1.6872	1.7891	2.2210	2.7666	2.6870
		b	0.8902	1.2657	1.9987	1.9005	2.5687	2.3027
600	Bias	a	0.0704	0.0940	0.1021	0.8450	0.8997	0.7002
		b	0.0231	0.1324	0.0450	0.2547	0.1239	0.0698
	MSE	a	0.5522	1.2103	1.3221	1.8542	2.0007	1.4641
		b	0.3012	1.0277	1.2318	1.5614	2.0021	1.9088
1000	Bias	a	0.0103	0.0440	0.0855	0.1538	0.2000	0.1002
		b	0.0096	0.0933	0.0152	0.1089	0.0966	0.0490
	MSE	a	0.0774	0.5541	0.8154	0.8012	1.0232	0.8971
		b	0.0746	0.6522	0.8231	0.7972	0.9981	0.9997

5. The ULE quantile regression model

As mentioned in the introduction of this article, Mitnik and Baek (2013) proposed the Kumaraswamy quantile regression model as an alternative to the beta regression model through the re-parameterization of the Kumaraswamy distribution in terms of its quantile. Since then, several authors have followed the same approach to propose various quantile regression models based on unit distributions. For example, Korkmaz and Chesneau (2021) introduced the unit Burr-XII quantile regression model, and Korkmaz and *et al.* (2022) proposed the unit Chen quantile regression model.

In this section, we propose an alternative quantile regression model by re-parameterizing the ULE distribution in terms of its quantile. The quantile function

of the ULE distribution is given by

$$Q(u) = \exp\left(\frac{1}{b}\left(a - \sqrt{a^2 - 2b\log u}\right)\right),$$

where $u \in (0,1)$. Setting $\mu = Q(u; a, b)$, it can be verified that

$$b = \frac{2a\log\mu - 2\ln u}{(\log\mu)^2}.$$

Consequently, the PDF and CDF of the re-parameterized ULE distribution can be expressed as

$$\begin{aligned} g(y; a, \mu) = y^{a-1}&\left(a - \frac{(2a\log\mu - 2\log u)\ \log y}{(\log\mu)^2}\right) \\ &\times \exp\left(-\frac{(a\log\mu - \log u)(\log y)^2}{(\log\mu)^2}\right), \end{aligned} \tag{5.9}$$

and

$$G(y; a, \mu) = y^a \exp\left(-\frac{a\log\mu - \log u}{(\log\mu)^2}(\log y)^2\right),$$

respectively, where $a > 0$ is a shape parameter and $\mu \in (0,1)$ is the quantile parameter corresponding to the fixed quantile level u.

For modeling the conditional median, we set $u = 0.5$. Figure 2 illustrates various shapes of the PDF in (5.9), demonstrating that the re-parameterized ULE distribution can exhibit unimodal, modified bathtub-shaped, decreasing, or increasing forms, depending on the parameter values.

Let $y_1, \ldots, y_n$ denote a sample of observations, where each y_i is an observation from $Y_i \sim \text{ULE}(a, \mu_i, u)$ for $i = 1, \ldots, n$, with unknown parameters a and μ_i. The parameter u is assumed known. The ULE regression model is defined through the link function

$$g(\mu_i) = \mathbf{x}_i^\top \boldsymbol{\eta},$$

where $\boldsymbol{\eta} = (\eta_0, \eta_1, \ldots, \eta_p)^\top$ is the unknown regression parameter vector, and $\mathbf{x}_i = (1, x_{i1}, x_{i2}, \ldots, x_{ip})^\top$ is the known covariate vector for the i-th observation. The link function $g(\cdot)$ connects the covariates to the conditional quantile of the response variable Y. Specifically, when $u = 0.5$, the model describes the conditional median of Y.

Because the ULE distribution is supported on the open interval $(0,1)$, an appropriate choice for the link function is the logit link:

$$g(\mu_i) = \text{logit}(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right).$$

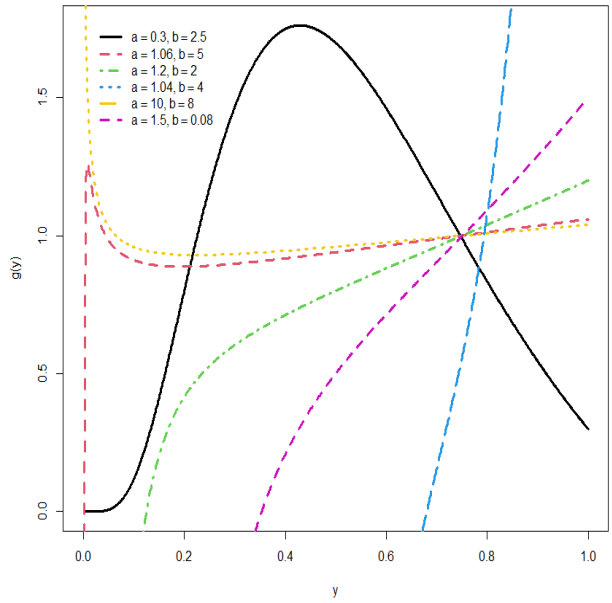

Figure 2: Some shapes of the PDF of the re-parameterized ULE distribution.

5.1 Estimation of the UC Quantile Regression Parameters

We estimate the unknown parameters of the ULE quantile regression model using the maximum likelihood estimation (MLE) technique. From the following link function

$$g(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right) = \mathbf{x}_i\boldsymbol{\eta}^\top,$$

we obtain the relation

$$\mu_i = \frac{\exp(\mathbf{x}_i\boldsymbol{\eta}^\top)}{1+\exp(\mathbf{x}_i\boldsymbol{\eta}^\top)}. \tag{5.10}$$

To obtain the MLE of $\boldsymbol{\xi} = (\boldsymbol{\eta}, a)^\top$, we maximize the following log-likelihood function of the ULE quantile regression model:

$$\begin{aligned}\ell(\boldsymbol{\xi}) = (a-1)\sum_{i=1}^{n}\log(y_i) \\ + \sum_{i=1}^{n}\log\left(a - \frac{2a\log(\mu_i) - 2\log u}{\log^2(\mu_i)}\log(y_i)\right) \\ - \sum_{i=1}^{n}\frac{a\log(\mu_i) - \log u}{\log^2(\mu_i)}\log^2(y_i),\end{aligned}$$

where μ_i is given by (5.10). The MLE of $\boldsymbol{\xi}$ was obtained using the `optim` function

in R. The Hessian matrix returned by optim was used to compute the asymptotic standard errors of the parameter estimates.

6. Data analysis application

6.1 Practical example for ULE model

To evaluate the flexibility and competency of our suggested ULE model, an application to a real data set is presented in this subsection. This data set was studied and reported by Caramanis and *et al.* (1983), where the objective was to compare the SC 16 and P3 algorithms to estimate the unit capacity factors. The data set analyzed here is derived from the P3 algorithm and consists of the following 22 observations: 0.853, 0.759, 0.866, 0.809, 0.717, 0.544, 0.492, 0.403, 0.344, 0.213, 0.116, 0.116, 0.092, 0.070, 0.059, 0.048, 0.036, 0.029, 0.021, 0.014, 0.011, 0.008 and 0.006.

To verify whether this data set can be appropriately modelled by the ULE distribution, we employ the Total Time on Test (TTT) plot, Aarset (1987). The TTT plot for the given data set is displayed in Figure 3, which exhibits a convex curve followed by a concave curve. This indicates that the data show a bathtub-shaped (U-shaped) HRF. Hence, the data set is efficiently modelled by the ULE distribution.

The fit of our introduced ULE distribution is compared with the fit of many new proposed models defined in the unit interval. The competitive models are:

- **Unit Rayleigh (UR)** distribution Bantan and *et al.* (2020) with PDF

$$f(x) = \frac{-2b\log x}{x}\exp\big(-b(\log x)^2\big), \quad x \in (0,1),\ b > 0.$$

- **Unit Extended Exponential (UExE)** distribution Ragab and *et al.* (2024) with PDF

$$f(x) = \frac{a^2(1-b\log x)}{b+a}\,x^{\,b-1}, \quad x \in (0,1),\ a,b > 0.$$

- **Unit Mirra (UM)** distribution Al-Omari and *et al.* (2024) with PDF

$$f(x) = \frac{a^3\big(b+(b+2)x^2-2bx\big)\,\exp\big(a-\frac{a}{x}\big)}{2x^4(a^2+b)}, \quad x \in (0,1),\ a,b > 0.$$

- **Unit Burr-XII (UBXII)** distribution Korkmaz and Chesneau (2021) with PDF

$$f(x) = ab\,x^{-1}(-\log x)^{b-1}\big(1+(-\log x)^b\big)^{-a-1}, \quad x \in (0,1),\ a,b > 0.$$

- **Exponentiated Unit Lindley (EUL)** distribution Irshad and *et al.* (2024) with PDF

$$f(x) = \left[1 - \frac{(b - x + 1)\exp\big(-\frac{bx}{1-x}\big)}{(b+1)(x-1)}\right]^{a-1} \cdot \frac{a\,b\,\exp\big(-\frac{bx}{1-x}\big)}{(1+b)(1-x)^3}, \quad x \in (0,1),\ a,b>0.$$

- **Bounded Weighted Exponential (BWE)** distribution Mallick and *et al.* (2021) with PDF

$$f(x) = \frac{b(a+1)}{a}\, x^{b-1}\big(1 - x^{ab}\big), \quad x \in (0,1),\ a,b>0.$$

- **Unit Exponential (UE)** distribution Bakouch and *et al.* (2023) with PDF

$$f(x) = \frac{2ab}{1-x^2}\left(\frac{1+x}{1-x}\right)^b \exp\left\{a\left[1 - \left(\frac{1+x}{1-x}\right)^b\right]\right\}, \quad x \in (0,1),\ a,b>0.$$

- **Unit-Weibull (UW)** distribution Mazucheli and *et al.* (2018) with PDF

$$f(x) = ab\,x^{-1}(-\log x)^{b-1}\exp\big(-a(-\log x)^b\big), \quad x \in (0,1),\ a,b>0.$$

- **Beta** distribution with PDF

$$f(x) = \frac{x^{b-1}(1-x)^{a-1}}{B(a,b)}, \quad x \in (0,1),\ a,b>0,$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$ is the beta function.

The comparison is based on several well-known model selection criteria, including minus twice the maximized log-likelihood ($-2\log L$), the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the corrected Akaike information criterion (AICc), the consistent Akaike information criterion (CAIC), and the Hannan-Quinn information criterion (HQC). Additionally, we employ the Kolmogorov-Smirnov (K-S), Cramér-von Mises (CvM), and Anderson-Darling (AD) goodness-of-fit statistics along with their corresponding p-values. The best-fitting model is selected as the one that simultaneously yields the largest p-values and the smallest values of $-2\log L$, AIC, BIC, AICc, CAIC, HQC, K–S, CvM, and AD statistics.

Using the R software, we compute the MLEs for all candidate models via the `mle2` function from the `bbmle` package. Table 3 presents the MLEs together with the values of $-2\log L$, AIC, BIC, AICc, CAIC, and HQC, while Table 4 reports the K-S, CvM, and AD statistics along with their p-values. The results in both tables indicate that the ULE distribution provides the best fit to the data set among all competing models. This conclusion is further corroborated by the visual assessment in Figure 4 , which confirms that the proposed distribution offers a superior fit to the empirical data.

Table 3: MLEs, AIC, BIC, AICc, CAIC and HQIC for data set.

Model	MLEs	-2logL	AIC	BIC	AICc	CAIC	HQIC
ULE	a=0.3042	-15.8096	-11.8096	-9.6275	-11.1780	-7.6275	-11.2955
	b=0.1325						
UR	a=0.3344	-5.4713	-3.4713	-2.3802	-3.2713	-1.3802	-3.2143
UExE	a=0.7797	-14.2969	-10.2969	-8.1148	-9.6653	-6.1148	-9.7828
	b=0.9442						
UM	a=0.0010	-6.4083	-2.4083	-0.2262	-1.7767	1.7738	-1.8942
	b=0.0751						
UBXII	a=0.8375	-6.8663	-2.8663	-0.6842	-2.2347	1.3158	-2.3522
	b=1.5218						
EUL	a=0.6181	-10.9495	-6.9495	-4.7674	-6.3179	-2.7674	-6.4354
	b=0.3409						
BWE	a=30.1194	-14.0543	-10.0543	-7.8722	-9.4227	-5.8722	-9.5402
	b=0.5201						
UE	α=26.0822	-10.5898	-6.5898	-4.4078	-5.9583	-2.4078	-6.0758
	b=0.0473						
UW	α=0.4003	-14.3510	-10.3510	-8.1689	-9.7194	-6.1689	-9.8369
	b=1.2259						
Beta	a=0.554	-13.5639	-9.5639	-7.3818	-8.9323	-5.3818	-9.0498
	b=1.220						

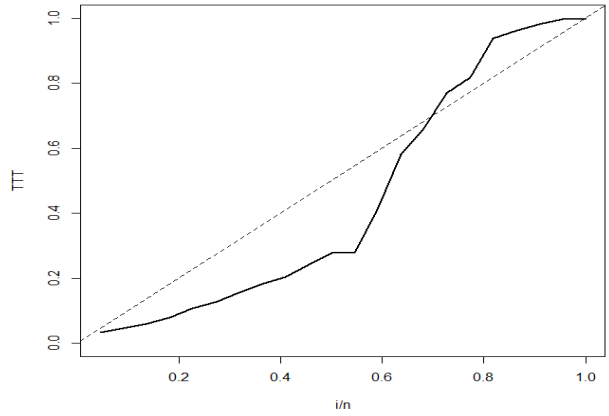

Figure 3: TTT plot of data set.

Table 4: K-S, CvM and AD statistics with their p-values.

Model	K-S	p-value	CvM	p-value	AD	p-value
ULE	0.15973	0.6285	0.0968	0.6050	0.6075	0.6390
UR	0.2423	0.1511	0.2763	0.1578	3.4444	0.0167
UExE	0.1841	0.4451	0.1063	0.5586	0.6255	0.6223
UMirra	0.2898	0.0497	0.4336	0.0581	4.0691	0.0083
UBXII	0.2434	0.1475	0.2141	0.2426	1.1667	0.2800
EUL	0.2350	0.1758	0.2194	0.2335	1.1671	0.2799
BWE	0.2055	0.3109	0.1303	0.4584	0.7281	0.5341
UE	0.2905	0.0488	0.3841	0.0788	2.3916	0.0571
UW	0.1833	0.4507	0.1208	0.4954	0.7512	0.5159
Beta	0.2002	0.3413	0.1264	0.4733	0.7241	0.5373

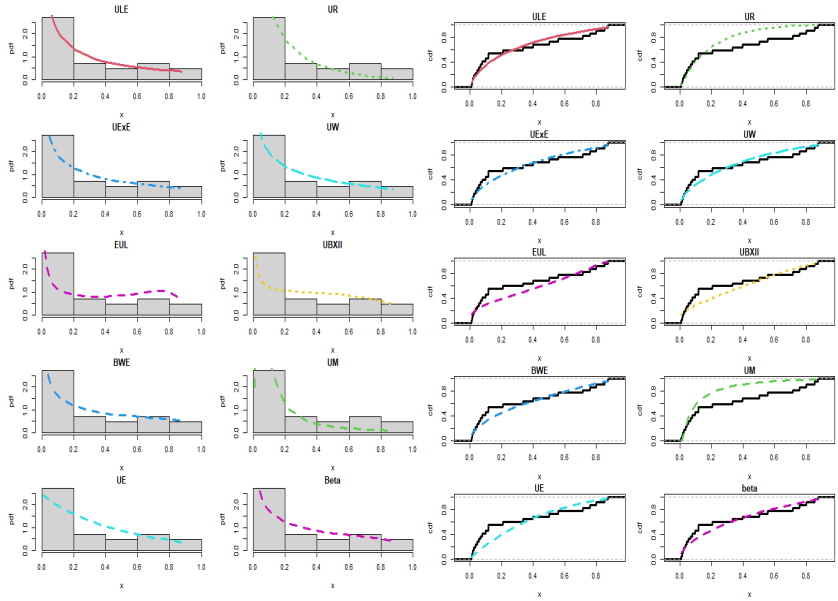

Figure 4: Plot of PDFs and CDFs of all fitted distributions.

6.2 Practical Example: ULE Quantile Regression Model

A real data set is provided to illustrate the superiority and practical utility of the ULE quantile regression model. This data set is taken from the `simplexreg` package in R software and concerns the recovery rate of CD34+ cells after peripheral blood stem cell transplants. The sample consists of 239 patients. This data set has been previously analyzed by several authors, like Mazucheli and *et al.* (2019),

Korkmaz and Chesneau (2021), and Korkmaz and *et al.* (2022). The response variable, y, represents the proportion of recovered CD34+ cells ($y \in (0,1)$), while the covariates associated with y are:

- x_1: gender (1 for male and 0 for female),
- x_2: chemotherapy protocol duration (0 for one day, 1 for three day),
- x_3: patient age (in years).

Due to the presence of observed outliers in the response variable, quantile regression is more appropriate. The ULE quantile regression model is compared with five well-known quantile regression models defined on $(0,1)$. The competitive models are:

- Beta regression Ferrari and Cribari-Neto (2004),
- Kumaraswamy (Kum) quantile regression model, Mitnik and Baek (2013),
- Unit Chen (UC) quantile regression model, Korkmaz and *et al.* (2022),
- Unit Burr-XII (UBXII) quantile regression model, Korkmaz and Chesneau (2021),
- Simplex regression model, Kieschnick and McCullough (2003).

The regression structure for all models is based on the logit link function:

$$\text{logit}(\mu_i) = \eta_0 + \eta_1 x_{i1} + \eta_2 x_{i2} + \eta_3 x_{i3}, \quad i = 1, \ldots, 239.$$

Table 5 reports the maximum likelihood estimates (MLEs), their standard errors (SEs), p-values, log-likelihood ($-L$), AIC, and BIC for all six models. From this table, it is evident that the ULE regression model has the lowest values of $-L$, AIC, and BIC compared to all competing models, indicating a superior model fit to the CD34+ recovery rate data. Also from this table, the parameter η_1 is not statistically significant for all regression models, suggesting that the gender variable does not significantly influence y. In contrast, the parameters η_2 (chemotherapy duration) and η_3 (age) are statistically significant at conventional levels (e.g., 0.05) in the ULE regression model. These results lead to the following interpretations:

- Each additional year of age is associated with an increase of 0.035 units in recovery time. This implies slower recovery in older patients,
- Longer chemotherapy duration is associated with delayed recovery; specifically, patients receiving a 3-day chemotherapy protocol experience significantly longer recovery times compared to those on a 1-day protocol.

Table 5: MLEs, S.E, p-values, AIC and BIC of all regression models.

Model	η_0	η_1	η_2	η_3	a	ℓ	AIC	BIC
ULE	2.8916	0.2328	0.6044	0.0351	0.7860	-197.4428	-384.8856	-367.5033
S.E	1.4335	0.1772	0.1840	0.0094	0.3221			
p-value	0.0224	0.0950	0.0006	0.0001	0.0077			
Beta	0.9990	0.0659	0.2116	0.0142	11.3450	-195.8956	-381.7912	-364.4089
S.E	0.1278	0.0939	0.1017	0.0053	1.0180			
p-value	<0.0001	0.4829	0.0374	0.0072	<0.0001			
Kum.	1.1997	0.0418	0.1833	0.0107	6.7274	-192.8299	-375.6599	-358.2775
S.E	0.1397	0.0955	0.1150	0.0059	0.4543			
p-value	<0.0001	0.6619	0.1123	0.0692	<0.0001			
UC	0.8987	0.1223	0.3131	0.0189	1.5272	-195.8254	-381.6508	-364.2685
S.E	0.1377	0.0985	0.1035	0.0053	0.0842			
p-value	<0.0001	0.1078	0.00139	0.0002	<0.0001			
UBXII	1.0428	0.0584	0.2248	0.0165	0.4135	-197.1325	-384.2649	-366.8826
S.E	0.1484	0.1060	0.1218	0.0063	1.0432			
p-value	<0.0001	0.5818	0.0649	0.0088	<0.0001			
Simplex	1.0683	0.0504	0.2645	0.0135	0.9275	156.7209	-303.4418	-303.1843
S.E	0.1606	0.1171	0.1245	0.0336	0.0458			
p-value	<0.0001	0.6667	0.0336	0.0380	<0.0001			

7. Conclusion

We have introduced a new unit distribution, termed the *unit linear exponential distribution*, derived via the transformation $X = e^{-Y}$, where Y follows the linear exponential distribution. The fundamental properties of this new distribution have been investigated. Model parameters were estimated using six classical estimation methods, and a simulation study demonstrated that all six estimators perform well. Also, we have proposed a novel quantile regression model based on this distribution, providing a flexible framework for modeling bounded response variables. The superior performance of the proposed distribution and regression model was validated through applications to two real-world data sets. In both cases, the new model yielded a significantly better fit compared to several existing competitive models.

Acknowledgment

The author would like to thank the editor and the anonymous reviewers for their insightful comments and constructive suggestions.

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