Journal of Data Science and Modeling, Vol. 2, No. 2, 149-165, June 2024

Research Manuscript

Some results on a generalized Archimedean copula with a stock market applicability

Shaghayegh Molaei[1],
Kianoush Fathi Vajargah[1,*], Hamid Mottaghi Golshan[2]

[1] Department of Statistics, North Tehran Branch, Islamic Azad University, Tehran, Iran,

[2] Department of Mathematics, Shahriar Branch, Islamic Azad University, Shahriar, Iran.

Recieved: 21/01/2025 Accepted: 20/04/2025

Abstract: This article examines the probability structure and dependency structure of a new family of Archimedean copula functions that are generated with two generators; this family is known as a generalization of the Archimedean copula functions and provides more tail dependence properties than the Archimedean family, making it more applicable. Using simulations, we compare a member of this family with various existing copula functions to highlight similarities and differences, and if the desired copula's scatter plot in terms of tail dependence is similar to the generalized Archimedean copula, we can fit the generalized Archimedean copula function to it.

Applications of this copula in the financial domain are demonstrated to improve the study of the dependence between indicators and to utilize this copula's advantageous characteristics. These theoretical concepts are validated by the numerical example provided at the end of the paper.

Keywords: Copula; Archimedean copula; Generalized Architecture copulas; Tail dependence.

Mathematics Subject Classification (2010): 62H20, 91G70.

*Corresponding Author: K_fathi@iau-tnb.ac.ir

1. Introduction

This study introduces a new family of generalized Archimedean copulas generated by combining two distinct generators. By enhancing tail dependence properties, this new family better captures the intricate dependencies present in multivariate data. The two-generator copula functions exhibit superior tail dependence properties. This means they can more accurately model extreme joint occurrences in datasets, which is crucial in fields like finance and risk management. This approach offers greater flexibility in modeling dependence structures. It can adapt to a wider range of data distributions, particularly those with strong dependencies that existing methods may not adequately capture.

The idea and broad definitions of copula, multivariate copula, and Archimedean copula are covered in the first section of the article. View the most recent works by Amblard and Girard (2002); Rodríguez-Lallena and Úbeda-Flores (2004); Nelsen (2006); Li (2013); Durante and Sempi (2016); Joe et al. (2010). The features of dependence and tail dependence are as follows; refer to Nelsen (2006); Luca and Rivieccio (2012); Joe et al. (2010); Genest et al. (2024). The second section introduces and studies a class of bivariate copulas, an extension of the well-known Archimedean family. The function induced is a copula, and it is investigated in the following under what circumstances the generators of this function Durante et al. (2007); Genest and Rivest (1993); Esary and Proschan (1972). Additionally, there exist other instances of extensions of this family of Archimedean copulas and their connections to other prominent Archimedean copulas; these may be found in Durante (2006); Genest and Rivest (1993); Wang and Wells (2000); Kasper (2024); Chesneau and Alhadlaq (2024); Guan and Wang (2024); Górecki and Okhrin (2024); Okhrin and Ristig (2024). The use of family associative generators to create generalized Archimedean copulas and the process by which one creates a generalized Archimedean copula by joining associative generators of Archimedean copulas are given below.

1.1 Copula

First, the definition of copula Schweizer and Wolff (1981); Durante and Sempi (2016) is given in formal terms. Let R stand for the ordinary real line $(-\infty, \infty)$, $\overline{R}^2$ for the extended real plane $\overline{R} \times \overline{R}$, and $\overline{R}$ for the extended real line $[-\infty, +\infty]$. The Cartesian product of two closed intervals is a rectangle $\overline{R}^2$: $B = [x_1, x_2] \times [y_1, y_2]$. The joint (x_1, y_1), (x_2, y_1), $(x_1, y_2), (x_2, y_2) \in B$. The product of $I \times I$, where $I = [0, 1]$, is the unit square I^2.

Definition 1.1. *A function $\mathcal{C}$ from I^2 with the following characteristics is called a copula:*

i For each s, t *in* I

$$\mathcal{C}(s,0) = 0 = \mathcal{C}(0,t), \qquad \textit{and} \qquad \mathcal{C}(s,1) = s, \mathcal{C}(1,t) = t. \tag{1.1}$$

ii For each set of values x_1, x_2, y_1, y_2 *in* I *such that* $x_1 \leq x_2$ *and* $y_1 \leq y_2$*,*

$$\mathcal{C}(x_2, y_2) - \mathcal{C}(x_2, y_1) - \mathcal{C}(x_1, y_2) + \mathcal{C}(x_1, y_1) \geq 0. \tag{1.2}$$

If $\mathcal{C}$ is a copula, then $\max(t+s-1,0) \leq \mathcal{C}(t,s) \leq \min(t,s)$ for any $(t,s) \in \mathrm{Dom}\mathcal{C}$. These are bounds, and they are usually expressed as $w(t,s) = \max(t+s-1,0)$ and $M(t,s) = \min(t,s)$.

We now revisit it in the bivariate context, as per the subsequent Theorem, with regard to A. Sklar, who created copulas Schweizer and Wolff (1981).

Theorem 1.2 (Sklar's Theorem)**.** *If* H *is a joint function of distribution having margins* F *and* G*, then for each* x, y *in* $\overline{\mathbb{R}}$*, there exists a copula* $\mathcal{C}$*, and if* F *and* G *are continuous, then*

$$H(x,y) = \mathcal{C}(F(x), G(y)), \tag{1.3}$$

is unique; if not, $\mathcal{C}$ *is unique on* $RangF \times RangG$*. Conversely, if* $\mathcal{C}$ *is a copula and* F, G *are distribution functions, then the function* H *given by* (1.3) *is a joint distribution function having margins* F *and* G *and*

$$\mathcal{C}(x,y) = H(F^{(-1)}(x), G^{(-1)}(y)). \tag{1.4}$$

Definition 1.3 (see Marshall et al. (1979))**.** *If* $c_1(x,y) \leq c_2(x,y)$ *for all* $x, y \in I$*, then* $c_1 \prec c_2$ *(or* $c_2 \succ c_1$*) is written when* c_1 *and* c_2 *are copulas. This suggests that either* c_2 *is* ***larger*** *than* c_1 *or* c_1 *is* ***smaller*** *than* c_2*.*

The copula product of two variables, x and y, is expressed as $\mathcal{C}(x,y) = \Pi(x,y) = uv$.

There are several methods for generating observations (x,y) of a pair of random variables (x,y) with a joint function of distribution H. In this section, we will focus on using the copula as a tool. By applying Sklar's theorem, we may use an approach similar to the one in the preceding paragraph to convert uniform variates. This is all that is needed to generate a pair (x,y) of observations of uniform (0,1) random variables (x,y) whose joint distribution function is $\mathcal{C}$, the copula of x and y. This process requires a conditionally distributed function for y with x, y, which we designate as $\mathcal{C}_x(y)$:

$$\mathcal{C}_x(y) = p[Y \leq y | X = x] = \lim_{\Delta x \to 0} \frac{\mathcal{C}(x+\Delta x, y) - \mathcal{C}(x,y)}{\Delta x} = \frac{\partial}{\partial x}\mathcal{C}(x,y).$$

Observe that practically everywhere in I, the function $y \to \frac{\partial}{\partial x}\mathcal{C}(x,y)$, which we now use to denote $\mathcal{C}_x(y)$, exists and is non-decreasing.

1. Create two uniform $(0,1)$ variates, x and t, that are independent.
2. Establish $y = \mathcal{C}_x^{(-1)}(t)$, where a quasi-inverse of $\mathcal{C}_x$ is indicated by $\mathcal{C}_x^{(-1)}$.
3. (x, y) is the desired pairing.

2. Bivariate Archimedean Copulas

Archimedean copulas have found widespread applications due to several appealing properties:

1. The ease with which they can be constructed;
2. The wide variety of copula families they encompass; and
3. The numerous elegant mathematical properties they exhibit.

Let φ be a strictly decreasing function from I to $[0, \infty]$ such that $\varphi(1) = 0$ and

$$\varphi(\mathcal{C}(x, y)) = \varphi(x) + \varphi(y). \tag{2.5}$$

Definition 2.1. *Let φ be a continuous, strictly decreasing function from I to $[0, \infty]$ such that $\varphi(1) = 0$. The pseudo-inverse of φ, denoted $\varphi^{[-1]}$, is defined as*

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \leq t \leq \varphi(0), \\ 0, & \varphi(0) \leq t \leq \infty. \end{cases} \tag{2.6}$$

The domain of $\varphi^{[-1]}$ is $[0, \infty]$, and its range is I.

Note that $\varphi^{[-1]}$ is continuous and non-increasing on $[0, \infty]$, and strictly decreasing on $[0, \varphi(0)]$. Furthermore, we have $\varphi^{[-1]}(\varphi(x)) = x$ for all $x \in I$, and

$$\varphi(\varphi^{[-1]}(t)) = \begin{cases} t, & 0 \leq t \leq \varphi(0), \\ 0, & \varphi(0) \leq t \leq \infty, \end{cases} = \min(t, \varphi(0)). \tag{2.7}$$

In the special case where $\varphi(0) = \infty$, we have $\varphi^{-1} = \varphi^{[-1]}$.

Lemma 2.2. *Assume φ is a strictly decreasing, continuous function from I to $[0, \infty]$, and let $\varphi^{[-1]}$ be its pseudo-inverse as defined in (2.6). Define the function $\mathcal{C} : I^2 \to I$ by*

$$\mathcal{C}(x, y) = \varphi^{[-1]}(\varphi(x) + \varphi(y)). \tag{2.8}$$

Then $\mathcal{C}$ satisfies the boundary conditions (1.1) and (1.2), and hence is a copula.

Theorem 2.3. *Let φ be a continuous, strictly decreasing function from I to $[0, \infty]$ with $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse defined in (2.6). The function $\mathcal{C} : I^2 \to I$ given by (2.8) is a copula if and only if φ is convex.*

The function φ is called the generator of the copula, and copulas of the form (2.8) are referred to as Archimedean copulas. If $\varphi(0) = \infty$, we say that φ is a strict generator, and in this case, both $\varphi^{[-1]}$ and $\mathcal{C}(x, y) = \varphi^{-1}(\varphi(x)+\varphi(y))$ define a strict Archimedean copula.

Theorem 2.4. *Let $\mathcal{C}$ be an Archimedean copula with generator φ. Then:*

1. *The following properties hold:*
   - *(a)* ***Symmetry:*** *$\mathcal{C}(x, y) = \mathcal{C}(y, x)$ for all $x, y \in I$;*
   - *(b)* ***Associativity:*** *$\mathcal{C}(\mathcal{C}(x, y), w) = \mathcal{C}(x, \mathcal{C}(y, w))$ for all $x, y, w \in I$;*
   - *(c) For any $c > 0$, $c\varphi$ is also a generator of $\mathcal{C}$.*
2. *Suppose $\mathcal{C}$ is an associative copula such that for every $x \in (0, 1)$, the function $\delta_c(x) < x$. Then $\mathcal{C}$ is an Archimedean copula.*
3. *Let $\mathcal{C}$ be an Archimedean copula generated by $\varphi \in \Phi$. Then there exists a positive integer n such that $x_c < y$ for any $x, y \in I$.*
4. *Every Archimedean copula has convex level curves.*

3. Tail Dependence

Definition 3.1. *Let (X, Y) be a vector with the marginal distribution functions F and G representing continuous random variables. The following*

$$\lambda_U = \lim_{u\to 1} P\{Y > G^{-1}(u)|X > F^{-1}(u)\}. \tag{3.9}$$

is the coefficient of upper tail dependence of (X, Y). Assume the existence of the limit $\lambda_U \in [0, 1]$. Furthermore, given that $\lim \lambda_L \in [0, 1]$ exists, the coefficient of lower tail dependence of (X, Y) is

$$\lambda_L = \lim_{u\to 0} F\{Y \le G^{-1}(t)|X \le F^{-1}(t)\}. \tag{3.10}$$

Theorem 3.2. *Given $\lambda_L, \lambda_U, F, G, X, Y$, let $\mathcal{C}$ be the copula of X, Y with diagonal section $\delta_{\mathcal{C}}$, then*

$$\lambda_U = 2 - \lim_{t\to 1} \frac{1 - \mathcal{C}(t, t)}{1 - t} = 2 - \delta'_{\mathcal{C}}(1^-), \tag{3.11}$$

exists if the limits (3.9) and (3.10) exist.

Proof. We have

$$\begin{aligned}\lambda_U &= \lim_{t\to 1^-} P[Y > G^{-1}(t)|X < F^{-1}(t)]\\ &= \lim_{t\to 1^-} \frac{\overline{\mathcal{C}}(t, t)}{1 - t} = \lim_{t\to 1^-} \frac{1 - 2t + \mathcal{C}(t, t)}{1 - t}\\ &= 2 - \lim_{t\to 1^-} \frac{1 - \mathcal{C}(t, t)}{1 - t} = 2 - \delta'_{\mathcal{C}}(1^-).\end{aligned}$$

For λ_L, the proof is the same. $\square$

If $\lambda_U \in (0,1)$, then the copula $\mathcal{C}$ has upper tail dependence; otherwise, it has upper tail independence.

Corollary 3.3. *If $\mathcal{C}$ is an Archimedean copula with generator $\varphi \in \Omega$, then*

$$\lambda_U = 2 - \lim_{t \to 1^-} \frac{1 - \varphi^{[-1]}(2\varphi(t))}{1-t} = 2 - \lim_{t \to 1^+} \frac{1 - \varphi^{[-1]}(2x)}{1 - \varphi^{[-1]}(x)},$$

and

$$\lambda_U = \lim_{t \to 0^+} \frac{\varphi^{[-1]}(2\varphi(t))}{t} = \lim_{x \to \infty} \frac{\varphi^{[-1]}(2x)}{\varphi^{[-1]}(x)}. \tag{3.12}$$

4. The generalized bivariate Archimedean copulas

Let Φ represent the class of all continuous and strictly decreasing functions $\varphi : [0,1] \to [0,\infty]$, and let ψ represent the class of all continuous, decreasing functions $\psi : [0,1] \to [0,\infty]$ with $\psi(1) = 0$. We also define $\mathit{\Phi}_0 = \mathit{\Phi} \cap \mathit{\Psi}$.

The function $\mathcal{C}_{\varphi,\psi} : [0,1]^2 \to [0,1]$ is defined by

$$\mathcal{C}_{\varphi,\psi}(x,y) = \varphi^{[-1]}(\varphi(x \wedge y) + \psi(x \vee y)), \tag{4.13}$$

where $x \wedge y = \min\{x,y\}$ and $x \vee y = \max\{x,y\}$. Keep in mind that all of these functions are symmetric, meaning that for any $x, y \in [0,1]$, $\mathcal{C}_{\varphi,\psi}(x,y) = \mathcal{C}_{\varphi,\psi}(y,x)$. For every $x \in [0,1]$,

$$0 \le \mathcal{C}_{\varphi,\psi}(x,0) = \varphi^{[-1]}(\varphi(0) + \psi(x)) \le \varphi^{[-1]}(\varphi(0)) = 0,$$

can be quickly shown using the equality $\varphi^{[-1]}(\varphi(t)) = t$. Also, $\mathcal{C}_{\varphi,\psi}$ satisfies (1.1) and (1.2), which are the boundary conditions.

Theorem 4.1 (Durante (2006))**.** *Let us assume that φ and ψ are members of $\mathit{\Phi}$ and $\mathit{\Psi}$, respectively. We define the function $\mathcal{C} = \mathcal{C}_{\varphi,\psi}$ by (4.13). Then, $\mathcal{C}$ is a copula if φ is convex and $(\psi - \varphi)$ is increasing on $[0,1]$. Since the function $t \mapsto (\psi(t) - \varphi(t))$ is increasing, it follows that for all $t \in [0,1]$, $\varphi(t) \ge \psi(t)$.*

Indeed, if $x_0 \in (0,1)$ and $\varphi(x_0) < \psi(x_0)$, then

$$0 < \psi(x_0) - \varphi(x_0) \le \psi(1) - \varphi(1) \le 0,$$

which is a contradiction.

Theorem 4.2. *Let h, k be two continuous, increasing functions from $[0,1]$ to $[0,1]$, with $k(1)=1$. If h is log-concave and $t \mapsto \frac{h(t)}{k(t)}$ is increasing, then*

$$\mathcal{C}_{h,k}(x,y) = h^{[-1]}(h(x \wedge y)k(x \vee y))$$

is a copula.

We now introduce a class of bivariate copulas that includes several well-known copulas. Let f be a function from $[0,1]$ to $[0,1]$. For all $u, v \in [0,1]$, the function $\mathcal{C}_f$ is defined as:

$$\mathcal{C}_f = (t \wedge s) f(t \vee s), \tag{4.14}$$

where $t \wedge s = \min(t,s)$ and $t \vee s = \max(t,s)$. It is evident that the function $\mathcal{C}_f$ is symmetric in the sense that for each $t, s \in [0,1]$, $\mathcal{C}_f(r,s) = \mathcal{C}_f(s,t)$, as shown in Durante (2006).

Theorem 4.3. *Consider a continuous function $f : [0,1] \to [0,1]$ that is differentiable everywhere except at a finite number of points. The function defined by equation (4.14) is denoted as $\mathcal{C}_f$. In this case, $\mathcal{C}_f$ is a copula if and only if the following conditions hold:*

1. $f(1) = 1$
2. *f is increasing*
3. *$t \mapsto \frac{f(t)}{t}$ is decreasing on the interval $[0,1]$.*

Example 4.4. *Assume that $f(t) = t - \ln t$ and that $\varphi(t) = -\ln t, \psi(t) = -\ln t(t - \ln t)$. According to Theorem 4.3, the function f is appropriate because:*

1. $f(1) = 1 - \ln(1) = 1$
2. *$f'(t) = 1 - \frac{1}{t}$ is increasing*
3. *$\frac{f(t)}{t} = 1 - \frac{\ln(t)}{t}$ is decreasing on $[0,1]$*

Thus, for all $t, s \in [0,1]$, we have:

$$\begin{aligned}
\mathcal{C}_{\varphi,\psi}(t,s) &= \varphi^{-1}(\varphi(t \wedge s) + \psi(t \vee s)) \\
&= \exp\left\{-\left(-\ln(t \wedge s) + (-\ln((t \wedge s) - \ln(t \vee s)))\right)\right\} \\
&= \exp\left\{\ln(t \wedge s) + \ln\left((t \wedge s) - \ln(t \vee s)\right)\right\} \\
&= (t \wedge s) f(t \vee s).
\end{aligned}$$

The copula function is of the type (2.8).

Example 4.5. *Let* $\psi(x) = -\ln x^{\alpha}, \alpha \in [0,1]$ *and* $\varphi(x) = -\ln x$. *Then for all* $t, s \in [0,1]$, *we have:*

$$\begin{aligned}\mathcal{C}_{\varphi,\psi}(t,s) &= \varphi^{-1}\left(-\ln(t \wedge s) + (-\ln(t \vee s)^{\alpha})\right) \\ &= \exp\left\{-\left(-\ln(t \wedge s) + (-\ln(t \wedge s)^{\alpha})\right)\right\} \\ &= (t \wedge s)(t \vee s)^{\alpha},\end{aligned}$$

and

$$\mathcal{C}_a(t,s) = \begin{cases}(t \wedge s)(t \vee s)^{\alpha}, & \textit{if } t \leq s, \\ t^{\alpha}s, & \textit{if } t > s.\end{cases} \tag{4.15}$$

$\mathcal{C}_a$ belongs to the copula family Cuadras-Auge. Theorem 4.3 states that $\mathcal{C}(t,s) = (t \wedge s)f(t \vee s)$ represents a copula, where f is a suitable function from $[0,1]$ to $[0,+\infty]$. Therefore, $\mathcal{C}$ is of the type (4.15). To determine this, we take $\psi(t) = -\ln f(t)$ and $\varphi(t) = -\ln t$. For all $t, s \in [0,1]$, we have:

$$\begin{aligned}\mathcal{C}_{\varphi,\psi}(t,s) &= \varphi^{-1}(\varphi(t \wedge s) + \psi(t \vee s)) \\ &= \varphi^{-1}\left(-\ln(t \wedge s) + (-\ln f(t \vee s))\right) \\ &= \exp\left\{-\left(-\ln(t \wedge s) + (-\ln f(t \vee s))\right)\right\} \\ &= (t \wedge s)f(t \vee s).\end{aligned}$$

The two-generator copula functions, which generalize the Archimedean copula family, exhibit superior tail dependence properties. This means they can more accurately model extreme joint occurrences in datasets, which is crucial in fields like finance and risk management.

5. Comparison of a member of the family of generalized Archimedean copula with different Archimedean Copulas

Building on the previously discussed results, we have discovered that the copula $\mathcal{C}_{\varphi,\psi}$ (from the generalized Archimedean copula class) can be constructed if φ and ψ are valid generators. The dependency structure properties of this class were only recently introduced, and as such, the copula function data containing members of this class may not always be easily identified in practice. This makes it difficult to infer their joint distribution solely from the dependence properties. Therefore, this section conducts an empirical investigation to address this issue. We generate data from the generalized Archimedean copula and perform hypothesis testing with several Archimedean copulas in an attempt to determine the degree of resemblance between these two distributions.

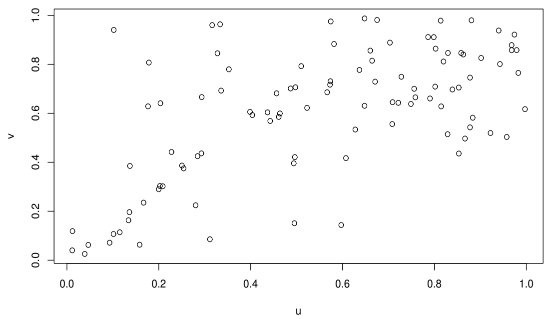

Figure 1: The scatter plot of data generated from the Archimedean copula

The generalized Archimedean copula is used to create data, while other Archimedean copulas are used to test hypotheses. The null hypothesis of the Archimedean copula has not been rejected, and we made an error in practice if the null hypothesis is true for any of the copulas other than the generalized Archimedean copula, which is not derived from the generalized Archimedean copula. With generator $\varphi(t) = \frac{1}{t} - 1, \psi(t) = -\ln t$, let us consider a desired generalized Archimedean copula. The copula function is as follows:

$$t = \frac{\partial \mathcal{C}(t,s)}{\partial t} = \begin{cases} \frac{1}{(1-t\ln s)^2}, & t \leq s, \\ \frac{1}{t(1-s\ln t)^2}, & t > s. \end{cases}$$

Then

$$\mathcal{C}_t^{(-1)}(t) = \begin{cases} \exp \frac{\sqrt{t}-1}{(t\sqrt{t})}, & t \leq s, \\ \frac{\sqrt{ut}}{1-\sqrt{ut}\ln t}, & t > s. \end{cases}$$

Figure 1 displays the scatter plot of the data produced by the Archimedean copula.

Figure 1 illustrates that while λ_U is almost equivalent to zero, λ_L is opposite to zero for these data and there is minimal reliance. This problem may also be demonstrated using the scatter plot and the following methods:

$$\begin{aligned} \lambda_U &= 2 - \lim_{t\to 1^-} \frac{1-\mathcal{C}(t,t)}{1-t} \\ &= 2 - \lim_{t\to 1^-} \frac{1-\varphi^{-1}(\varphi(t,t)+\psi(t,t))}{1-t} \\ &= 2 - \lim_{t\to 1^-} \frac{1-\ln t - t}{(1-t)(1-t\ln t)} \\ &= 2 - \lim_{t\to 1^-} \frac{-\ln t - 1 - 1}{-(1-t\ln t)+(-\ln t-1)(1-t)} = 2-2=0, \end{aligned}$$

and

$$\begin{aligned}\lambda_L &= \lim_{t\to 0^+}\frac{\mathcal{C}(t,t)}{t}\\ &= \lim_{t\to 0^+}\frac{t}{t(1-t\ln t)}\\ &= \lim_{t\to 1^-}\frac{1}{(1-t\ln t)+t(-\ln t-1)} = 1 \neq 0.\end{aligned}$$

We now take into consideration those Archimedean families that, like this Archimedean copula, are generalized in terms of upper and lower tail dependency. To what degree the chosen Archimedean Copulas and the generalized Archimedean copulas differ is our goal. The following are the chosen copulas (refer to (Nelsen, 2006, 4.2.12, 4.2.16, 4.2.19 and 4.2.20)):

i

$$\mathcal{C}_\varphi(x,y) = \left(1+\left[(x^{-1}-1)^\theta+(y^{-1}-1)\right]^{\frac{1}{\theta}}\right)^{-1},$$

where

$$\varphi(t) = \left(\frac{1}{t}-1\right)^\theta, \theta\in[1,\infty], \lambda_L = 2^{-\frac{1}{\theta}}, \lambda_U = 2-2^{\frac{1}{\theta}}.$$

ii

$$\mathcal{C}_\varphi(x,y) = \frac{1}{2}\left(S+\sqrt{S^2+4\theta}\right), S = x+y-1-\theta\left(\frac{1}{x}+\frac{1}{y}-1\right),$$

where

$$x,y\in[0,1], \varphi(t) = \left(\frac{\theta}{t}+1\right)(1-t), \theta\in[0,\infty], \lambda_L = \frac{1}{2}, \lambda_U = 0.$$

iii

$$\mathcal{C}_\varphi(x,y) = \frac{\theta}{\ln\left(e^{\theta/x}+e^{\theta/y}-e^\theta\right)},$$

where

$$x,y\in[0,1], \varphi(t) = e^{\frac{\theta}{t}}-e^\theta, \theta\in(0,\infty), \lambda_L = 1, \lambda_U = 0.$$

iv

$$\mathcal{C}_\varphi(x,y) = \left[\ln(\exp(x^{-\theta})+\exp(y^{-\theta})-e)\right]^{-1/\theta},$$

where

$$x,y\in[0,1], \varphi(t) = \exp(t^{-\theta}-e), \theta\in(0,\infty), \lambda_L = 1, \lambda_U = 0.$$

The Archimedean copulas i, ii, iii, and iv are shown as scatter plots in Figures 2. Density function graphs in three dimensions are shown in Figures 3. The study primarily focuses on hypothesis testing to check if the generalized Archimedean copula can be distinguished from traditional Archimedean copulas. Table 1 demon-

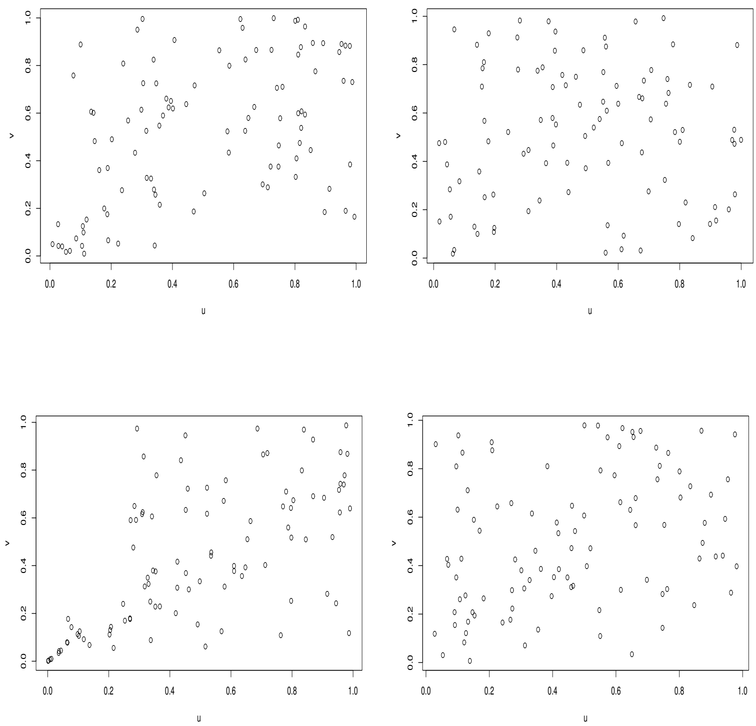

Figure 2: The scatter plot of data generated from copulas i, ii, iii and iv, respectively which illustrates the relationship between random variables generated through the copula

Generated from GAC	Copula i	Copula ii	Copula iii	Copula iv
Probability of type I error	0.356	0.408	0.996	0.251

Table 1: The data are generated from the generalized Archimedean copula

Generate data	Copula i	Copula ii	Copula iii	Copula iv
Probability of type I error	0.23	0.704	0.008	0.582

Table 2: The data are generated from copulas i, ii, iii and iv

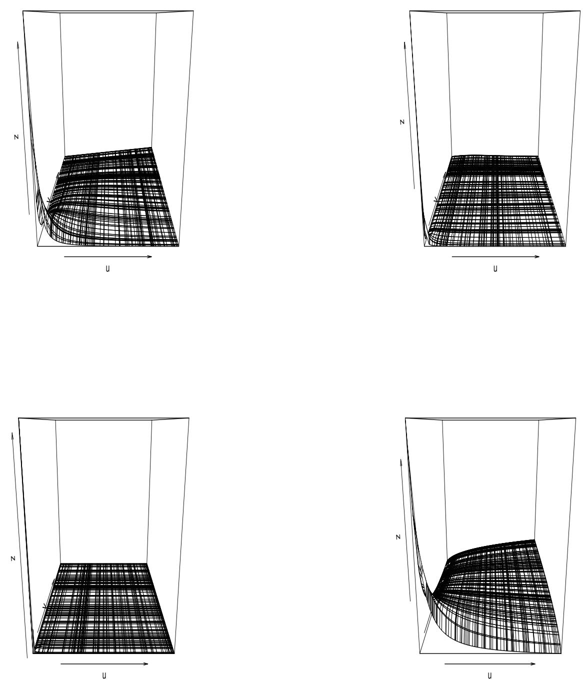

Figure 3: Three-dimensional graphs of density function copulas i, ii, iii and iv, respectively. Each scatter plot demonstrates varying degrees of upper and lower tail dependence, highlighting how the generalized Archimedean copula can better model extreme dependencies in financial scenarios

strates that the null hypothesis of copulas i, ii, iii, and iv is not rejected at the 0.05 level when the data are created from the generalized Archimedean copula.

Table 2 demonstrates that the null hypothesis of the generalized copula is not rejected at the 0.05 level when the data are created from copulas i, ii, iii, and iv.

Stated differently, Tables 1 and 2 demonstrate that, in accordance with the tail dependence figure, it is impossible to accurately distinguish between the generalized Archimedean copula and the copulas i, ii, iii, and iv, as well as the copulas i, ii, iii, and iv, as well as the generalized Archimedean copula.

5.1 Practical example

The second-largest stock exchange globally and the world's first electronic stock market is the Nasdaq Stock Exchange, or NASDAQ for short. The National Association of Securities Dealers (NASD) created this prominent financial market in February 1971 with the intention of offering a quicker and simpler method for buying and selling stocks. More than 3,000 reputable companies, including some of the biggest names in technology, including Apple, Amazon, and Facebook, have now moved their shares to the Nasdaq and are listed there. Initially focusing on over-the-counter stocks and providing services such as the automatic collection of the latest stock price information, Nasdaq gradually grew, and today it is recognized as one of the most important and leading stock markets in the world. One of its historical milestones is the launch of the first stock exchange website and the first online transactions, which shows Nasdaq's pioneering use of new technologies. The Nasdaq stock market is highly secure due to the use of advanced electronic systems for trading, although it has faced security challenges in the past, like any other system. However, it is still considered one of the safest financial markets in the world, where companies large and small can offer their shares to the public and investors can easily trade them.

The Nasdaq stock market has various indicators that are used to evaluate the performance of companies in this market. Its two most important indices include the Nasdaq Composite Index and the Nasdaq-100 Index.

The Nasdaq Composite Index is one of the most important stock market indices that shows the overall performance of the Nasdaq market.

The NASDAQ Composite includes all domestic and foreign companies that are traded on the Nasdaq market, and the impact of companies on this index is determined by their market capitalization. Therefore, the price movements of larger companies have a greater impact on the index.

In July 2024, the index's daily data were acquired. We can determine whether or not the Archimedean copula function is appropriate for fitting this data since we may connect several variables, some of whose distributions may not be known,

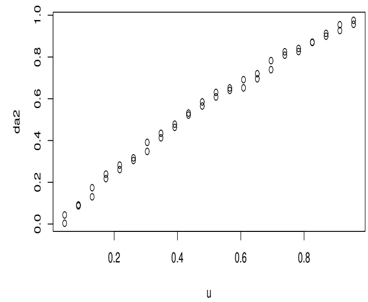

Figure 4: **Scatter plot of NASDAQ data with a generalized Archimedean copula**

using the copula function.
Based on the scatter plot, we can infer that the data distribution is nearly the same if the data are positioned on the first and third bisectors. Figure 4 illustrates how the GAC can be more effectively applied and analyzed when data have tail dependences similar to those of the generalized Archimedean copula function. This is particularly useful when analyzing various financial markets to examine the relationships between indicators and their effects on one another. The generalized Archimedean copula function that best matches the data can be used.
Thus, the generalized Archimedean Copula function that provides a good fit to the data can be used.

The analysis of the NASDAQ Composite Index highlights the dependency structures among major technology stocks, providing crucial insights for investors. Our findings reveal significant tail dependence, with an upper tail dependence coefficient λ_U estimated at approximately 0.7, indicating a noteworthy likelihood of joint upward movements during bullish phases. This understanding is essential for risk management and strategic asset allocation, allowing investors to capitalize on correlated movements effectively.

6. Conclusion

The generalized Archimedean copula model offers a robust framework for analyzing dependencies among financial assets, making it a valuable tool for risk assessment and portfolio optimization. By effectively capturing tail dependencies and allowing for nuanced evaluations of asset interactions, this model can significantly enhance financial decision-making processes, ultimately leading to better risk management, optimized portfolios, and more accurate pricing of complex financial instruments.

Numerous uses of Archimedean copulas have been found, including: 1) their simplicity of construction; 2) the huge variety of copula families that belong to the class; and 3) the numerous pleasant features that each member of the class possesses. We may fit the generalized Archimedean copula function to the desired copula if, in terms of tail dependency, its scatter plot resembles that of the copula, to build suitable joint distribution functions for a better description of the dependence between the available information and a better understanding of the impact of each of them, especially in the financial fields and analysis of indicators.

Declarations

Funding

None.

Availability of data and materials

The data that support the findings of this study are openly available at https://www.marketwatch.com/investing/index/comp/download-data.

Competing interests

The authors declare that they have no competing interests.

Contributions

All authors contributed equally. All authors read and approved the final manuscript.

Acknowledgements

We would like to thank to respected reviewers for their thorough reading of the manuscript and their helpful remarks that helped us to improve the manuscript.

References

Amblard, C. and Girard, S. (2002), Symmetry and dependence properties within a semiparametric family of bivariate copulas. *Journal of Nonparametric Statistics*, **14(6)**, 715-727.

Chesneau, C. and Alhadlaq, W. M. (2024), Exploring a new two-parameter Archimedean copula: the Gumbel-Joe copula. *Hacet. J. Math. Stat.*, **53(6)**, 1742-1758.

Durante, F. (2006), A new class of symmetric bivariate copulas. *Nonparametric Statistics*, **18(7-8)**, 499-510.

Durante, F. and Sempi, C. (2016), *Principles of copula theory.* CRC Press, Boca Raton, FL.

Durante, F., Quesada-Molina, J. J., and Sempi, C. (2007), A generalization of the Archimedean class of bivariate copulas. *Annals of the Institute of Statistical Mathematics*, **59(3)**, 487-498.

Esary, J. D. and Proschan, F. (1972), Relationships among some concepts of bivariate dependence. *The Annals of Mathematical Statistics*, **43(2)**, 651-655.

Genest, C. and Rivest, L.-P. (1993), Statistical inference procedures for bivariate Archimedean copulas. *Journal of the American Statistical Association*, **88(423)**, 1034-1043.

Genest, C., Okhrin, O., and Bodnar, T. (2024), Preface to the special issue "Copula modeling from Abe Sklar to the present day". *J. Multivariate Anal.*, 201, Paper No. 105280, 1.

Górecki, J. and Okhrin, O. (2024), *Hierarchical Archimedean copulas.* SpringerBriefs in Applied Statistics and Econometrics. Springer, Cham.

Guan, Q. and Wang, B. X. (2024), Some properties of convex and increasing convex orders under Archimedean copula. *Probab. Engrg. Inform. Sci.*, **38(4)**, 632-643.

Joe, H., Li, H., and Nikoloulopoulos, A. K. (2010), Tail dependence functions and vine copulas. *J. Multivariate Anal.*, **101(1)**, 252-270.

Kasper, T. M. (2024), On convergence and singularity of conditional copulas of multivariate Archimedean copulas, and conditional dependence. *J. Multivariate Anal.*, 201, Paper No. 105275, 14.

Li, H. (2013), Toward a copula theory for multivariate regular variation. In *Copulae in mathematical and quantitative finance*, volume 213 of *Lect. Notes Stat.*, 177-199. Springer, Heidelberg.

Luca, G. D. and Rivieccio, G. (2012), Multivariate tail dependence coefficients for Archimedean copulae. In *Advanced statistical methods for the analysis of large data-sets*, 287-296. Springer.

Marshall, A. W., Olkin, I., and Arnold, B. C. (1979), *Inequalities: theory of majorization and its applications*, volume 143. Springer.

Nelsen, R. B. (2006), Archimedean copulas. *An Introduction to Copulas*, 109-155, springer.

Okhrin, O. and Ristig, A. (2024), Penalized estimation of hierarchical Archimedean copula. *J. Multivariate Anal.*, 201, Paper No. 105274, 16.

Rodríguez-Lallena, J. A. and Úbeda-Flores, M. (2004), A new class of bivariate copulas. *Statistics & Probability Letters*, **66(3)**, 315-325.

Schweizer, B. and Wolff, E. F. (1981), On nonparametric measures of dependence for random variables. *The Annals of Statistics*, **9(4)**, 879-885.

Wang, W. and Wells, M. T. (2000), Model selection and semiparametric inference for bivariate failure-time data. *Journal of the American Statistical Association*, **95(449)**, 62-72.