Journal of Data Science and Modeling, Vol. 2, No. 1, 185-208, December 2023

Research Manuscript

The Exponentiated New XLindley Distribution: Properties, and Applications

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Received: 29/07/2024 Accepted: 02/11/2024

Abstract: In this paper, we introduce a new extension of the XLindley distribution, called the Exponentiated New XLindley Distribution. The new model has an increasing or bathtub-shaped hazard rate function, making it suitable for modeling real-life phenomena. We study important properties of the new model, such as moments, the moment-generating function, incomplete moments, mean deviations from the mean and the median, Bonferroni and Lorenz curves, the mean residual life function, Rényi entropy, order statistics, and k-record values. We also address the estimation of parameters using maximum likelihood and bootstrap methods. A Monte Carlo simulation study is conducted to evaluate the estimators discussed in the paper. Additionally, we analyze two real data applications, including rainfall and COVID-19 data sets, to demonstrate the applicability and flexibility of the new distribution. Our results show that the new model fits the data sets better than several other recognized or recently introduced distributions, based on several well-known goodness-of-fit criteria.

Keywords: Bootstrap estimation, Exponentiation method, Maximum likelihood estimation, Moments, New XLindley distribution, Simulation.

Mathematics Subject Classification (2010): 60E05, 62F10, 62F12.

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1. Introduction

Introducing new distributions has attracted the interest and attention of many statisticians in recent years. The key goal is to find new, more flexible distributions that can model real data in many fields, such as engineering, management, economics, health sciences, and reliability, more accurately. There are many methods to generalize a distribution and arrive at a potentially flexible model to fit specific data sets related to a real phenomenon; see, for example, Lee *et al.* (2013). One of the popular methods for extending a distribution is the exponentiation technique, which involves adding an extra parameter by exponentiating the cumulative distribution function (CDF) of the base distribution. Many authors have used the exponentiation procedure to develop new models; for example, Mudholkar and Srivastava (1993) introduced the exponentiated Weibull distribution, Nadarajah *et al.* (2011) introduced the exponentiated Lindley distribution, Warahena-Liyanage and Pararai (2014) and Ashour and Eltehiwy (2015) introduced the exponentiated power Lindley distribution, Pourdarvish *et al.* (2015) introduced the exponentiated Topp-Leone distribution, Abdollahi Nanvapisheh *et al.* (2019) and Jayakumar and Elangovan (2019) introduced the exponentiated Shanker distribution, and Alomair *et al.* (2024) introduced the exponentiated XLindley distribution.

Recently, Khodja *et al.* (2023) introduced a new one-parameter lifetime distribution, called the new XLindley distribution, whose probability density function (PDF) is given as follows:

$$f_{NX}(x) = \frac{\theta}{2}(1+\theta x)\exp(-\theta x), \qquad x, , \theta > 0. \tag{1.1}$$

The new XLindley distribution with PDF (1.1) is a mixture of the exponential distribution with parameter θ and the gamma distribution with parameters 2 and θ; in other words, we have

$$f_{NX}(x) = \frac{1}{2}f_1(x) + \frac{1}{2}f_2(x),$$

where $f_1(x)$ and $f_2(x)$ are the PDFs of the exponential distribution with parameter θ and the gamma distribution with parameters 2 and θ.

The CDF of the new XLindley distribution is given by

$$F_{NX}(x) = 1 - \big(1 + \frac{\theta x}{2}\big)\exp(-\theta x), \qquad x, , \theta > 0. \tag{1.2}$$

Khodja *et al.* (2023) proved that the new XLindley distribution possesses a decreasing PDF and an increasing hazard rate function (HRF). They also presented various properties of the new XLindley distribution. The goal of this paper is to introduce a new two-parameter distribution, called the Exponentiated New XLindley (ENXL) distribution, that can be established by the exponentiation method based on (1.2). The new model has an increasing and bathtub-shaped HRF, which makes it very suitable for fitting a wide variety of lifetime data sets. In fact, the ENXL distribution can work better than many other lifetime distributions for modeling lifetime phenomena.

The rest of the paper is organized as follows: In Section 2, the new model is proposed, and some of its properties, such as moments, incomplete moments, mean deviations from the mean and the median, Bonferroni and Lorenz curves, the mean residual life function, Rényi entropy, order statistics, and k-record values, are discussed. The problem of estimating the parameters using the maximum likelihood (ML) and bootstrap methods is investigated in Section 3. A simulation study is presented in Section 4 to examine the performance of the point and interval estimators. Section 5 is devoted to two real data applications. The results of Section 5 confirm that the new model is well-suited for modeling real phenomena. Several remarks conclude the paper.

2. The New Model and Some of Its Properties

Let the random variable X have an ENXL distribution with parameter θ. Then, the CDF of X is given by

$$F(x) = \left[1 - \left(1 + \frac{\theta x}{2}\right)\exp(-\theta x)\right]^{\alpha}, \qquad x > 0,\ \alpha,\ \theta > 0. \tag{2.3}$$

Upon differentiating (2.3), we arrive at the PDF of X, given by

$$f(x) = \frac{\alpha\,\theta}{2}(1 + \theta x)\exp(-\theta x)\left[1 - \left(1 + \frac{\theta x}{2}\right)\exp(-\theta x)\right]^{\alpha - 1}, \qquad x > 0. \tag{2.4}$$

We $X \sim \text{ENXL}(\alpha, \theta)$ if the PDF of X can be written as (2.4). The HRF of the new model is also given by

$$h(x) = \frac{\alpha\,\theta(1 + \theta x)\exp(-\theta x)\left[1 - \left(1 + \frac{\theta x}{2}\right)\exp(-\theta x)\right]^{\alpha - 1}}{2\left(1 - \left[1 - \left(1 + \frac{\theta x}{2}\right)\exp(-\theta x)\right]^{\alpha}\right)}, \quad x > 0. \tag{2.5}$$

In Figure 1, we plotted the PDFs of the ENXL distribution for selected values of the parameters. It is noticeable that the shape of the PDF can be either decreasing or unimodal contingent on the values of the parameters. The plots of the HRFs of the ENXL distribution for selected values of the parameters are also given in Figure 2. Figure 2 reveals that the HRF is increasing or bathtub-shaped.

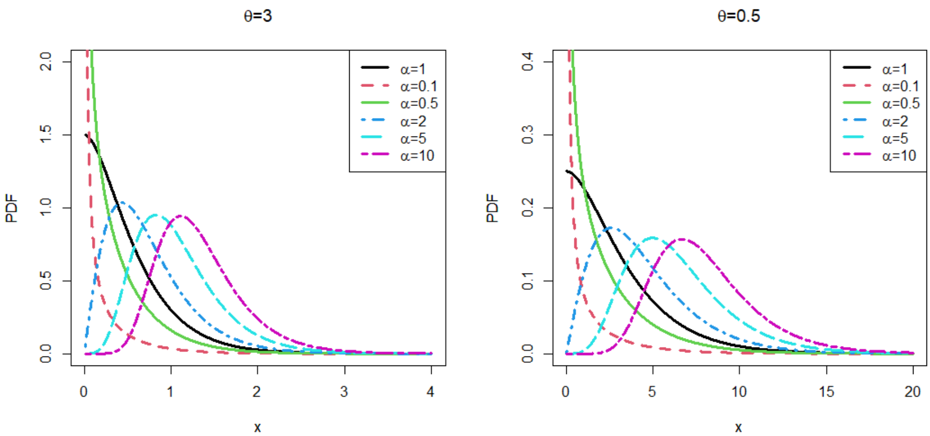

Figure 1: PDFs of the ENXL distribution for selected values of α and θ.

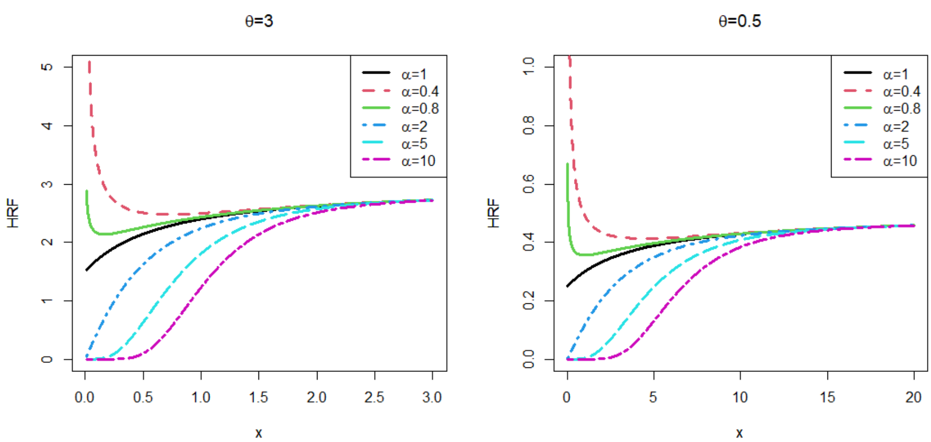

Figure 2: HRFs of the ENXL distribution for selected values of α and θ.

2.1 The Moments and Incomplete Moments

Moments are statistical measures that help describe the shape, central tendency, and variability of a distribution. In this subsection, the complete and incomplete moments of the ENXL distribution are studied. First, we obtain an expansion for $f(x)$. Consider the following equation, which is the generalized binomial expansion

$$(1-u)^a = \sum_{j=0}^{\infty} \binom{a}{j} (-1)^j \, u^j, \qquad |u| < 1.$$

Thus, the PDF of the ENXL can be expanded as follows

$$\begin{aligned} f(x) &= \frac{\alpha\,\theta}{2}(1+\theta x)\exp(-\theta x)\left[1-(1+\frac{\theta x}{2})\exp(-\theta x)\right]^{\alpha-1} \\ &= \frac{\alpha\,\theta}{2}(1+\theta x)\exp(-\theta x)\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\left(1+\frac{\theta x}{2}\right)^j \mathrm{e}^{-j\theta x} \\ &= \alpha\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{\alpha-1}{j}\binom{j}{s}\frac{(-1)^j\theta^{s+1}\,x^s}{2^{s+1}}(1+\theta x)\mathrm{e}^{-(j+1)\theta x}. \end{aligned} \tag{2.6}$$

Similarly, the expansion of the CDF of the new distribution is

$$\begin{aligned} F(x) &= \left[1-(1+\frac{\theta x}{2})\exp(-\theta x)\right]^{\alpha} = \sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^j\left(1+\frac{\theta x}{2}\right)^j \mathrm{e}^{-j\theta x} \\ &= \sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{\alpha}{j}\binom{j}{s}\frac{(-1)^j\theta^s\,x^s}{2^s}\mathrm{e}^{-j\theta x}. \end{aligned} \tag{2.7}$$

Now, using (2.6), the moment generating function of the ENXL distribution is obtained to be

$$M_X(t) = \alpha\sum_{j=0}^{\infty}\sum_{s=0}^{j}\frac{\binom{\alpha-1}{j}\binom{j}{s}(-1)^j\theta^{s+1}s!}{2^{s+1}[(j+1)\theta-t]^{s+1}}\left(1+\frac{(s+1)\,\theta}{(j+1)\theta-t}\right),\quad t<\theta.$$

Moreover, the r-th moment of X from (2.6) is given by

$$\mu_r = \alpha\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{\alpha-1}{j}\binom{j}{s}\frac{(-1)^j\Gamma(s+r+1)}{2^{s+1}(j+1)^{r+s+1}\theta^r}\left(1+\frac{s+r+1}{j+1}\right). \tag{2.8}$$

Therefore, the mean of the ENXL distribution is given by

$$\mu = \mu_1 = \alpha\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{\alpha-1}{j}\binom{j}{s}\frac{(-1)^j\Gamma(s+2)}{2^{s+1}(j+1)^{s+2}\theta}\left(1+\frac{s+2}{j+1}\right).$$

So, the skewness and kurtosis of the ENXL distribution are given by

$$S = \frac{E\left[(X-E(X))^3\right]}{\left(E\left[(X-E(X))^2\right]\right)^{3/2}} = \frac{\mu_3-3\mu_1\mu_2+2\mu_1^3}{[\mu_2-\mu_1^2]^{3/2}},$$

and

$$K = \frac{E\left[(X-E(X))^4\right]}{\left(E\left[(X-E(X))^2\right]\right)^{2}} = \frac{\mu_4-4\mu_1\mu_3+6\mu_1^2\mu_2-3\mu_1^4}{[\mu_2-\mu_1^2]^{2}},$$

respectively, where μ_r is given in (2.8).

Now, suppose that X is a random variable with a PDF given in (2.4). Let us obtain the r-th incomplete moment of X, which is given as follows

$$\begin{aligned}
\int_0^t x^r\, f(x)\,\mathrm{d}x &= \int_0^t x^r\, \frac{\alpha\,\theta}{2}(1+\theta x)\exp(-\theta x)\left[1-\left(1+\frac{\theta x}{2}\right)\exp(-\theta x)\right]^{\alpha-1}\mathrm{d}x\\
&= \alpha \sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{\alpha-1}{j}\binom{j}{s}\frac{(-1)^j\theta^{s+1}}{2^{s+1}}\\
&\quad\times\left[\int_0^t x^{r+s}\mathrm{e}^{-(j+1)\theta x}\mathrm{d}x+\int_0^t \theta x^{r+s+1}\mathrm{e}^{-(j+1)\theta x}\mathrm{d}x\right]\\
&= \alpha \sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty}\binom{\alpha-1}{j}\binom{j}{s}\frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\,t^{r+s+p+1}}{2^{s+1}\,p!}\\
&\quad\times\left[\frac{1}{r+s+p+1}+\frac{\theta\,t}{r+s+p+2}\right]. \qquad (2.9)
\end{aligned}$$

2.2 Mean Deviations from the Mean and Median

Let X be an ENXL-distributed random variable with PDF (2.4). Then, using (2.7) and (2.9), the mean deviation from the mean is given by

$$\begin{aligned}
\delta_1(X) &= \int_0^\infty |x-\mu|\, f(x)\,\mathrm{d}x = 2\mu\, F(\mu) - 2I(\mu)\\
&= 2\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{\alpha}{j}\binom{j}{s}\frac{(-1)^j\theta^s\,\mu^{s+1}}{2^s}\mathrm{e}^{-j\theta\mu}\\
&\quad -2\alpha \sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty}\binom{\alpha-1}{j}\binom{j}{s}\frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\,\mu^{s+p+2}}{2^{s+1}\,p!}\\
&\quad\times\left[\frac{1}{s+p+2}+\frac{\theta\,\mu}{s+p+3}\right],
\end{aligned}$$

where $I(b) = \int_0^b x\, f(x)\,\mathrm{d}x$ and $\mu = E(X)$.

Let M denote the median. Then, the mean deviation from the median is similarly obtained as follows:

$$\begin{aligned}
\delta_2(X) &= \int_0^\infty |x-M|\, f(x)\,\mathrm{d}x = \mu - 2I(M)\\
&= \mu - 2\alpha \sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty}\binom{\alpha-1}{j}\binom{j}{s}\frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\,M^{s+p+2}}{2^{s+1}\,p!}\\
&\quad\times\left[\frac{1}{s+p+2}+\frac{\theta\,M}{s+p+3}\right].
\end{aligned}$$

2.3 Bonferroni and Lorenz Curves

Here, we focus on the formulas of Bonferroni and Lorenz curves. These curves are used in various fields, such as economics, reliability, medicine, and insurance. Note that for $|z| < 1$ and $\rho > 0$, we have

$$(1-z)^{-\rho} = \sum_{q=0}^{\infty} \frac{\Gamma(\rho+q)\, z^q}{\Gamma(\rho)\, q!}. \tag{2.10}$$

Using (2.9) and (2.10), the Bonferroni curve can be derived as follows:

$$\begin{aligned}
B_F\,[F(x)] &= \frac{1}{\mu F(x)} \int_0^x u\, f(u)\, \mathrm{d}u = \frac{\alpha}{\mu\left[1-\left(1+\frac{\theta x}{2}\right)\exp(-\theta x)\right]^{\alpha}} \\
&\quad \times \sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty} \binom{\alpha-1}{j}\binom{j}{s} \frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\, x^{s+p+2}}{2^{s+1}\, p!} \\
&\quad \times \left[\frac{1}{s+p+2} + \frac{\theta\, x}{s+p+3}\right] \\
&= \frac{\alpha}{\mu} \sum_{q=0}^{\infty}\sum_{v=0}^{q}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty} \binom{\alpha-1}{j}\binom{j}{s}\binom{q}{v} \frac{\Gamma(\alpha+q)(-1)^{j+p}\,\theta^{v+s+1+p}}{\Gamma(\alpha)\, 2^{v+s+1}\, p!\, q!} \\
&\quad \times (j+1)^p x^{v+s+p+2} \mathrm{e}^{-q\,\theta\, x} \left[\frac{1}{s+p+2} + \frac{\theta\, x}{s+p+3}\right].
\end{aligned}$$

The Lorenz curve is also given by

$$\begin{aligned}
L_F\,[F(x)] &= \frac{1}{\mu} \int_0^x u\, f(u)\, \mathrm{d}u \\
&= \frac{\alpha}{\mu} \sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty} \binom{\alpha-1}{j}\binom{j}{s} \frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\, x^{s+p+2}}{2^{s+1}\, p!} \\
&\quad \times \left[\frac{1}{s+p+2} + \frac{\theta\, x}{s+p+3}\right].
\end{aligned}$$

The Lorenz and Bonferroni curves of the ENXL distribution for selected values of α and θ are displayed in Figure 3.

In economics, if $F(q)$ represents the proportion of units with incomes at or below q, then $L_F[F(q)]$ represents the proportion of total income accumulated by the set of units with an income at or below q. Similarly, the Bonferroni curve $B_F[F(q)]$ measures the mean income of this group relative to the mean income of the population (see Al-Shomrani and Al-Arfaj (2024)).

2.4 Mean Residual Life Function

The mean residual life (MRL) is a concept in survival analysis and reliability theory that quantifies the expected remaining lifetime of an object given that it

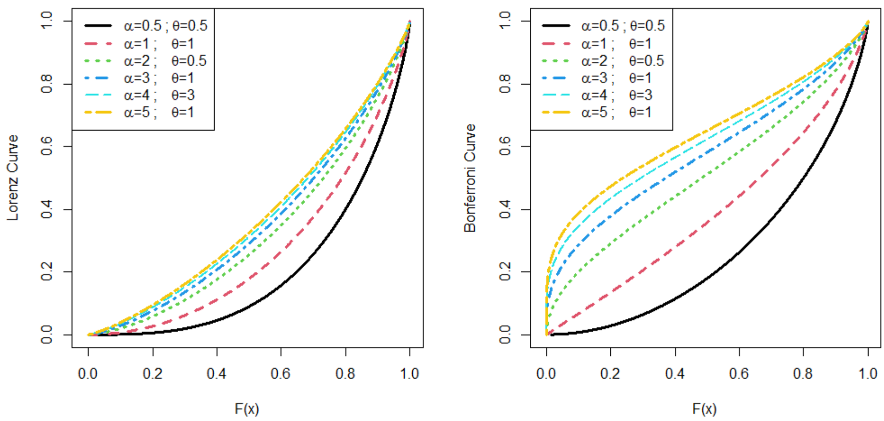

Figure 3: Lorenz (left panel) and Bonferroni (right panel) curves of the ENXL distribution for selected values of α and θ.

has survived up to a certain point in time, t. The MRL quantifies the expected remaining lifetime $X - t$, given that the lifetime X exceeds t. Thus, for $X \sim \text{ENXL}(\alpha, \theta)$, using (2.7) and (2.9), the MRL function is given by

$$\begin{aligned}
\mu(t) = E(X-t|X>t) &= \frac{1}{1-F(t)}\left(\mu - \int_0^t x f(x)\mathrm{d}x\right) - t \\
&= \left(\mu - \int_0^t x f(x)\mathrm{d}x\right)\sum_{q=0}^{\infty}[F(t)]^q - t \\
&= \left(\mu - \alpha \sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{p=0}^{\infty}\binom{\alpha-1}{j}\binom{j}{s}\frac{(-1)^{j+p}\,\theta^{s+1+p}\,(j+1)^p\,t^{s+p+2}}{2^{s+1}\,p!}\right. \\
&\quad \left.\times\left[\frac{1}{s+p+2}+\frac{\theta\,t}{s+p+3}\right]\right)\sum_{q=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{v}\binom{\alpha q}{v}\binom{v}{w}\frac{(-1)^v\theta^w\,x^w}{2^w}\mathrm{e}^{-v\theta x} - t.
\end{aligned}$$

2.5 Rényi Entropy

Entropy and information are commonly used to quantify uncertainty in a probability distribution. Additionally, numerous relationships have been derived based on the properties of entropy. The entropy associated with a random variable X measures the degree of uncertainty in its distribution. The Rényi entropy is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left[\int_0^{\infty}[f(x)]^{\gamma}\mathrm{d}x\right],$$

where $\gamma > 0$ and $\gamma \neq 1$.

Now, we have

$$\begin{aligned}
\int_0^\infty [f(x)]^\gamma \mathrm{d}x &= \int_0^\infty \frac{\alpha^\gamma\,\theta^\gamma}{2^\gamma}(1+\theta x)^\gamma \exp(-\theta\gamma x)\left[1-\left(1+\frac{\theta x}{2}\right)\exp(-\theta x)\right]^{\gamma(\alpha-1)}\mathrm{d}x \\
&= (\alpha\theta)^\gamma \sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{\gamma(\alpha-1)}{j}\frac{(-1)^j\binom{j}{s}}{2^{s+\gamma}}\int_0^\infty (\theta x)^s(1+\theta x)^\gamma \mathrm{e}^{-(j+\gamma)\theta x}\mathrm{d}x \\
&= \alpha^\gamma\theta^{\gamma-1}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{\gamma(\alpha-1)}{j}\frac{(-1)^j\binom{j}{s}}{2^{s+\gamma}}\int_1^\infty (u-1)^s u^\gamma \mathrm{e}^{-(j+\gamma)(u-1)}\mathrm{d}u \\
&= \alpha^\gamma\theta^{\gamma-1}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{i=0}^{s}\frac{\binom{\gamma(\alpha-1)}{j}\binom{j}{s}\binom{s}{i}(-1)^{j+s-i}}{2^{s+\gamma}\mathrm{e}^{-(j+\gamma)}}\int_1^\infty u^{i+\gamma}\mathrm{e}^{-(j+\gamma)u}\mathrm{d}u \\
&= \alpha^\gamma\theta^{\gamma-1}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{i=0}^{s}\frac{\binom{\gamma(\alpha-1)}{j}\binom{j}{s}\binom{s}{i}(-1)^{j+s-i}}{2^{s+\gamma}\mathrm{e}^{-(j+\gamma)}} \\
&\quad\times\left(\frac{\Gamma(i+\gamma+1)}{(j+\gamma)^{i+\gamma+1}}-\sum_{p=0}^{\infty}\frac{(-1)^p(j+\gamma)^p}{p!\,(i+\gamma+p+1)}\right).
\end{aligned}$$

Thus, we can express

$$\begin{aligned}
I_R(\gamma) = \frac{1}{1-\gamma}\Bigg(\log\Bigg[&\sum_{j=0}^{\infty}\sum_{s=0}^{j}\sum_{i=0}^{s}\frac{\binom{\gamma(\alpha-1)}{j}\binom{j}{s}\binom{s}{i}(-1)^{j+s-i}}{2^{s+\gamma}\mathrm{e}^{-(j+\gamma)}} \\
&\times\left\{\frac{\Gamma(i+\gamma+1)}{(j+\gamma)^{i+\gamma+1}}-\sum_{p=0}^{\infty}\frac{(-1)^p(j+\gamma)^p}{p!\,(i+\gamma+p+1)}\right\}\Bigg]+\gamma\log\alpha+(\gamma-1)\log\theta\Bigg).
\end{aligned}$$

2.6 Order Statistics

Order statistics play a crucial role in reliability analysis. Consider a random sample consisting of $X_1, X_2, \cdots, X_n$ drawn from an ENXL distribution. Let $X_{i:n}$ represent the i-th order statistic from this sample, and let $f_{i:n}(x)$ denote its PDF. Our goal is to derive a linear expansion for $f_{i:n}(x)$. Note that (see, for example, Arnold *et al.* (2008))

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!}{(i-1)!\,(n-i)!}\,f(x)\,F^{i-1}(x)\,\{1-F(x)\}^{n-i} \\
&= \frac{n!}{(i-1)!\,(n-i)!}\sum_{q=0}^{n-i}(-1)^q\binom{n-i}{q}f(x)\,F(x)^{q+i-1} \\
&= \sum_{q=0}^{n-i}a_{i,n,q}\frac{(q+i)\alpha\,\theta}{2}(1+\theta x)\mathrm{e}^{-\theta x}\left[1-\left(1+\frac{\theta x}{2}\right)\exp(-\theta x)\right]^{(q+i)\alpha-1}, \quad (2.11)
\end{aligned}$$

where $a_{i,n,q} = \dfrac{(-1)^q\,n!}{(i-1)!\,(n-i-q)!\,q!\,(q+i)}$.

From (2.11), we can see that the PDF of $X_{i:n}$ can be written as a finite combination of the ENXL densities. Therefore, many properties of these order statistics can be derived using this result. For example, from (2.8) and (2.11), the r-th moment of $X_{i:n}$ can be expressed as

$$E(X_{i:n}^r) = \alpha \sum_{q=0}^{n-i} \sum_{j=0}^{\infty} \sum_{s=0}^{j} \binom{(i+q)\alpha - 1}{j} \binom{j}{s} \frac{a_{i,n,q}(i+q)(-1)^j \Gamma(s+r+1)}{2^{s+1}(j+1)^{r+s+1}\theta^r} \times \left(1 + \frac{s+r+1}{j+1}\right).$$

2.7 k-Record Values

Record data are also important in real-life phenomena. An upper (lower) k-record process can be defined as the k-th largest (smallest) observation yet seen, see Arnold *et al.* (1998) and Ahmadi *et al.* (2012). Suppose that there exists a sequence of independent and identically distributed random variables that come from an ENXL distribution. Let $U_{m(k)}$ represent the m-th upper k-record value extracted from the mentioned sequence, and let $f_{m(k)}(x)$ denote its PDF. We have (see, for example, Arnold *et al.* (1998))

$$\begin{aligned} f_{m(k)}(x) &= \frac{k^m}{(m-1)!} f(x) \left[-\ln(1-F(x))\right]^{m-1} \{1-F(x)\}^{k-1} \\ &= \frac{k^m}{(m-1)!} \left[-\ln(1-F(x))\right]^{m-1} \sum_{q=0}^{k-1} (-1)^q \binom{k-1}{q} f(x) F(x)^q. \end{aligned} \tag{2.12}$$

Now, based on the MacLaurin expansion, we have

$$[-\ln(1-F(x))]^{m-1} = \left(F(x) \sum_{w=0}^{\infty} \frac{F(x)^w}{w+1}\right)^{m-1} = F(x)^{m-1} \left(\sum_{w=0}^{\infty} \frac{F(x)^w}{w+1}\right)^{m-1}. \tag{2.13}$$

Set $a_w = \frac{1}{w+1}$, then (see Gradshteyn and Ryzhik (2007), Equation 0.314)

$$\left(\sum_{w=0}^{\infty} a_w F(x)^w\right)^{m-1} = \sum_{w=0}^{\infty} c_w F(x)^w, \tag{2.14}$$

where the coefficients c_w for $w \geq 1$ are obtained from the following recurrence equation (with $c_0 = a_0^{m-1} = 1$)

$$c_w = \frac{1}{w} \sum_{z=1}^{w} \frac{(mz-w)c_{w-z}}{z+1}, \quad w \geq 1.$$

Now, from (2.12), (2.13), and (2.14), we may write

$$\begin{aligned} f_{m(k)}(x) &= \frac{k^m}{(m-1)!}\sum_{q=0}^{k-1}\sum_{w=0}^{\infty} c_w\,(-1)^q\binom{k-1}{q} f(x)\,F(x)^{m-1+w+q} \\ &= \sum_{q=0}^{k-1}\sum_{w=0}^{\infty} b_{w,m,q,k}\frac{(q+m+w)\alpha\,\theta}{2}(1+\theta x)\mathrm{e}^{-\theta x} \\ &\quad\times\left[1-\big(1+\frac{\theta x}{2}\big)\exp(-\theta x)\right]^{(q+m+w)\alpha-1}, \end{aligned} \tag{2.15}$$

where $b_{w,m,q,k} = \dfrac{k^m\,c_w\,(-1)^q\binom{k-1}{q}}{(m-1)!\,(q+m+w)}$.

From (2.15), we can see that the PDF of $U_{m(k)}$ can be expressed as an infinite combination of the ENXL densities. Similarly, we can show that the PDF of m-th lower k-record value can be written as an infinite combination of the ENXL densities, as well. Therefore, many properties of k-record values can be obtained based on this outcome. For instance, from (2.8) and (2.15), the r-th moment of $U_{m(k)}$ can be written as

$$E(U^r_{m(k)}) = \alpha\sum_{q=0}^{k-1}\sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\sum_{s=0}^{j}\binom{(q+m+w)\alpha-1}{j}\binom{j}{s}\frac{b_{w,m,q,k}(q+m+w)(-1)^j}{2^{s+1}(j+1)^{r+s+1}\theta^r} \times\Gamma(s+r+1)\left(1+\frac{s+r+1}{j+1}\right).$$

3. Parameter Estimation

Let $\boldsymbol{X} = (X_1,\cdots,X_n)$ denote a random sample of n from $X\sim$ ENXL(α,θ) and $\boldsymbol{x} = (x_1,\cdots,x_n)$ be the observed set of $\boldsymbol{X}$. In what follows, we study maximum likelihood and bootstrap methods to find point and interval estimators for the parameters of the ENXL distribution.

3.1 Maximum Likelihood Estimation

The likelihood function of the parameters given the random sample $\boldsymbol{x}$ is given by

$$\mathcal{L}(\alpha,\theta|\boldsymbol{x}) = \left(\frac{\alpha\,\theta}{2}\right)^n \exp\big(-\theta\sum_{i=1}^n x_i\big)\prod_{i=1}^n(1+\theta x_i)\left[1-\big(1+\frac{\theta x_i}{2}\big)\mathrm{e}^{-\theta x_i}\right]^{\alpha-1}. \tag{3.16}$$

Thus, the log-likelihood function becomes

$$\begin{aligned} \ell(\alpha,\theta|\boldsymbol{x}) &= n\log\left(\frac{\alpha\,\theta}{2}\right)-\theta\sum_{i=1}^n x_i+\sum_{i=1}^n\ln(1+\theta x_i) \\ &\quad+(\alpha-1)\sum_{i=1}^n\ln\left[1-\big(1+\frac{\theta x_i}{2}\big)\mathrm{e}^{-\theta x_i}\right]. \end{aligned} \tag{3.17}$$

Upon taking the derivatives of (3.17), we arrive at the following equations, which may help us find the ML estimates of the parameters:

$$\begin{aligned}\frac{\partial \ell(\alpha,\theta|\boldsymbol{x})}{\partial \alpha} &= \frac{n}{\alpha}+\sum_{i=1}^{n}\log\left[1-\big(1+\frac{\theta x_i}{2}\big)\mathrm{e}^{-\theta x_i}\right]=0,\\ \frac{\partial \ell(\alpha,\theta|\boldsymbol{x})}{\partial \theta} &= \frac{n}{\theta}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{x_i}{1+\theta x_i}+(\alpha-1)\sum_{i=1}^{n}\frac{x_i\,\mathrm{e}^{-\theta x_i}(1+\theta x_i)}{2[1-\big(1+\frac{\theta x_i}{2}\big)\mathrm{e}^{-\theta x_i}]}=0.\end{aligned}$$

Numerical methods can be utilized to solve the equations mentioned above.

Now, let $\widehat{\alpha}_M$ and $\widehat{\theta}_M$ denote the ML estimators of α and θ, respectively. Note that under certain conditions stated in Newey and McFadden (1994), the ML estimators are consistent. Moreover, under certain regularity conditions outlined in Lehmann and Casella (1998), as n approaches infinity, it follows that:

$\sqrt{n}\left((\widehat{\alpha}_M-\alpha,\widehat{\theta}_M-\theta)^T\right)\xrightarrow{D}N\left((0,0)^T,I_{X_1}^{-1}(\alpha,\theta)\right)$, where $\xrightarrow{D}$ denotes the convergence in distribution and $I_{X_1}^{-1}(\theta,\alpha)$ is the inverse matrix of $I_{X_1}(\theta,\alpha)$ in which $I_{X_1}(\theta,\alpha)$ is the Fisher information matrix of the parameters based on X_1, that is defined as

$$I_{X_1}(\alpha,\theta)=-\begin{bmatrix}E\Big(\frac{\partial^2\ln f(X_1;\alpha,\theta)}{\partial\alpha^2}\Big) & E\Big(\frac{\partial^2\ln f(X_1;\alpha,\theta)}{\partial\alpha\partial\theta}\Big)\\ E\Big(\frac{\partial^2\ln f(X_1;\alpha,\theta)}{\partial\theta\partial\alpha}\Big) & E\Big(\frac{\partial^2\ln f(X_1;\alpha,\theta)}{\partial\theta^2}\Big)\end{bmatrix}=\begin{bmatrix}I_{\alpha\alpha} & I_{\alpha\theta}\\ I_{\theta\alpha} & I_{\theta\theta}\end{bmatrix},$$

where $f(x;\alpha,\theta)\equiv f(x)$ is given in (2.4).

Now, $I_{X_1}(\theta,\alpha)$ may be estimated using the following relations

$$\begin{aligned}\widehat{I}_{\alpha\alpha}&=-\frac{1}{n}\frac{\partial^2\ell(\alpha,\theta|\boldsymbol{X})}{\partial\alpha^2}\bigg|_{(\alpha,\theta)=(\widehat{\alpha}_M,\widehat{\theta}_M)}=\frac{1}{\widehat{\alpha}_M^2},\\ \widehat{I}_{\alpha\theta}=\widehat{I}_{\theta\alpha}&=-\frac{1}{n}\frac{\partial^2\ell(\alpha,\theta|\boldsymbol{X})}{\partial\alpha\,\partial\theta}\bigg|_{(\alpha,\theta)=(\widehat{\alpha}_M,\widehat{\theta}_M)}=-\frac{1}{n}\sum_{i=1}^{n}\frac{X_i\,\mathrm{e}^{-\widehat{\theta}_M X_i}(1+\widehat{\theta}_M X_i)}{2[1-\big(1+\frac{\widehat{\theta}_M X_i}{2}\big)\mathrm{e}^{-\widehat{\theta}_M X_i}]},\\ \widehat{I}_{\theta\theta}&=-\frac{1}{n}\frac{\partial^2\ell(\alpha,\theta|\boldsymbol{X})}{\partial\theta^2}\bigg|_{(\alpha,\theta)=(\widehat{\alpha}_M,\widehat{\theta}_M)}=\frac{1}{\widehat{\theta}_M^2}+\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{1+\widehat{\theta}_M X_i}\right)^2\\ &\quad+\frac{\widehat{\alpha}_M-1}{n}\sum_{i=1}^{n}\frac{X_i^2\,\mathrm{e}^{-\widehat{\theta}_M X_i}(2\widehat{\theta}_M X_i+\mathrm{e}^{-\widehat{\theta}_M X_i})}{4[1-\big(1+\frac{\widehat{\theta}_M X_i}{2}\big)\mathrm{e}^{-\widehat{\theta}_M X_i}]^2},\end{aligned}$$

Now, we may find $100(1-\tau)\%$ two-sided asymptotic confidence intervals for the α and θ using the above-explained property of asymptotic normality of the ML estimators as follows

$$\widehat{\alpha}_M\pm z_{\frac{\tau}{2}}\frac{\widehat{\sigma}_1}{\sqrt{n}},\qquad\text{and}\qquad \widehat{\theta}_M\pm z_{\frac{\tau}{2}}\frac{\widehat{\sigma}_2}{\sqrt{n}}$$

respectively, where z_γ is the upper γ-th quantile of the standard normal distribution,

$$\widehat{\sigma}_1 = \sqrt{\frac{\widehat{I}_{\theta\theta}}{\widehat{I}_{\alpha\alpha}\widehat{I}_{\theta\theta} - \widehat{I}^2_{\alpha\theta}}}, \quad \text{and} \quad \widehat{\sigma}_2 = \sqrt{\frac{\widehat{I}_{\alpha\alpha}}{\widehat{I}_{\alpha\alpha}\widehat{I}_{\theta\theta} - \widehat{I}^2_{\alpha\theta}}}. \tag{3.18}$$

However, since the lower bounds can potentially take on negative values, we suggest using the delta method (see for example, Shao (2003)), which guarantees that the lower bounds will not get negative. Let $\varphi(\alpha) = \log\alpha$. Using the delta method, we have $\sqrt{n}(\varphi(\widehat{\alpha}_M) - \varphi(\alpha)) \xrightarrow{D} N\left(0, \frac{Var(\widehat{\alpha}_M)}{\alpha^2}\right)$. Therefore, a $100(1-\tau)\%$ asymptotic confidence interval (ACI) for $\varphi(\alpha)$ is given by

$$\varphi(\widehat{\alpha}_M) \pm z_{\frac{\tau}{2}} \frac{\widehat{\sigma}_1}{\widehat{\alpha}_M} \equiv (L_\alpha, U_\alpha).$$

So, the $100(1-\tau)\%$ ACI for α is given by

$$(\mathrm{e}^{L_\alpha}, \mathrm{e}^{U_\alpha}) \equiv \left(\widehat{\alpha}_M \exp\left\{-z_{\frac{\tau}{2}} \frac{\widehat{\sigma}_1}{\widehat{\alpha}_M}\right\}, \widehat{\alpha}_M \exp\left\{z_{\frac{\tau}{2}} \frac{\widehat{\sigma}_1}{\widehat{\alpha}_M}\right\}\right),$$

where $\widehat{\sigma}_1$ is given in (3.18).

Similarly, the $100(1-\tau)\%$ ACI for θ is given by

$$\left(\widehat{\theta}_M \exp\left\{-z_{\frac{\tau}{2}} \frac{\widehat{\sigma}_2}{\widehat{\theta}_M}\right\}, \widehat{\theta}_M \exp\left\{z_{\frac{\tau}{2}} \frac{\widehat{\sigma}_2}{\widehat{\theta}_M}\right\}\right),$$

where $\widehat{\sigma}_2$ is given in (3.18).

3.2 Parametric Bootstrap Estimation

In this subsection, we consider the parametric bootstrap method to find point and interval estimators for the parameters. For details regarding bootstrap methods, one can refer to Efron (1982) and Davison and Hinkley (1997). The following algorithm is employed to generate parametric bootstrap samples.

Algorithm 3.1.

Step 1: Calculate the ML estimates of α *and* θ, *denoted by* $\hat{\alpha}_M$ *and* $\hat{\theta}_M$, *respectively, based on* $\boldsymbol{x}$.

Step 2: Generate the bootstrap sample $X_1^*, \ldots, X_n^*$, *from* $\mathrm{ENXL}(\hat{\alpha}_M, \hat{\theta}_M)$.

Step 3: Calculate the ML estimates of α *and* θ *based on the generated bootstrap sample in Step 2, denoted by* $\hat{\alpha}^*$ *and* $\hat{\theta}^*$, *respectively.*

Step 4: Perform Steps 2 and 3, B times, and record $(\hat{\alpha}_i^*, \hat{\theta}_i^*)$ *for* $i = 1, \ldots, B$, *which can be represented as the set* $\{(\hat{\alpha}_1^*, \hat{\theta}_1^*), \cdots, (\hat{\alpha}_B^*, \hat{\theta}_B^*)\}$.

Now, sensible point estimates for α and θ can be given by

$$\widehat{\alpha}_B = \frac{1}{B}\sum_{i=1}^{B}\hat{\alpha}_i^*, \qquad \text{and} \qquad \widehat{\theta}_B = \frac{1}{B}\sum_{i=1}^{B}\hat{\theta}_i^*,$$

respectively.

Next, we propose two bootstrap-type confidence intervals, percentile bootstrap confidence interval (PBCI) and basic bootstrap confidence interval (BBCI). Sort the values of $\hat{\alpha}_i^*$'s in increasing order and denote the i-th value in the ordered set as $\hat{\theta}^{(i)}$ for $i = 1, \cdots, B$. Then, the $100(1-\tau)\%$ PBCI and the $100(1-\tau)\%$ BBCI for α are given by

$$\left(\hat{\alpha}^*_{((B+1)\frac{\tau}{2})}, \hat{\alpha}^*_{((B+1)(1-\frac{\tau}{2}))}\right), \quad \text{and} \quad \left(2\hat{\alpha}_M - \hat{\alpha}^*_{((B+1)(1-\frac{\tau}{2}))}, 2\hat{\alpha}_M - \hat{\alpha}^*_{((B+1)\frac{\tau}{2})}\right),$$

respectively, where $\hat{\alpha}_M$ is the ML estimate of α.

We may find the $100(1-\tau)\%$ PBCI and $100(1-\tau)\%$ BBCI for θ similarly.

4. A Simulation Study

In this section, we conduct a Monte Carlo simulation study to evaluate the performance of the proposed estimators. We select the parameter values as $(\alpha, \theta) = (2, 1), , (3, 1.5)$ and $(2, 2)$. The sample sizes are chosen to be $n = 25, 40$ and 60, with the simulation being replicated $M = 2000$ times. We calculate the ML and bootstrap point estimates, as well as the 95% ACIs, PBCIs, and BBCIs. Let $\widehat{\alpha}$ be an estimator of α and $\widehat{\alpha}i$ be the corresponding estimate derived in the i-th replication. Then, the estimated bias (EB) and the estimated mean squared error (EMSE) of $\widehat{\alpha}$ are given by

$$\text{EB}(\widehat{\alpha}) = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\alpha}_i - \alpha), \quad \text{and} \quad \text{EMSE}(\widehat{\alpha}) = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\alpha}_i - \alpha)^2,$$

respectively.

We may define the above criteria for the point estimators of θ analogously. To assess and compare the estimators, we calculate the ABs and EMSEs of the point estimators, along with the average widths (AWs) and coverage probabilities (CPs) of the 95% interval estimators, and the results are presented in Tables 1 and 2.

From Tables 1 and 2, we observe the superiority of ML estimators over the bootstrap ones. Besides, the 95% ACIs based on the delta method exhibit AWs that are less than the ones related to the 95% bootstrap confidence intervals. We may state that most of the CPs are close to the nominal value of 0.95. Moreover, the EMSEs, EBs, and AWs decrease as the sample size increases, as expected.

Table 1: The EMSEs (and the EBs in the parentheses) of the point estimators.

	$(\alpha, \theta) = (2, 1)$			
	α		θ	
n	ML	Bootstrap	ML	Bootstrap
25	0.8801 (0.3077)	1.8587 (0.7266)	0.0498 (0.0630)	0.0682 (0.1327)
40	0.3523 (0.1897)	0.5822 (0.4047)	0.0273 (0.0384)	0.0339 (0.0798)
60	0.1962 (0.1132)	0.2763 (0.2426)	0.0179 (0.0278)	0.0208 (0.0548)
	$(\alpha, \theta) = (3, 1.5)$			
	α		θ	
n	ML	Bootstrap	ML	Bootstrap
25	2.8370 (0.5590)	6.8757 (1.3631)	0.0971 (0.0811)	0.1311 (0.1776)
40	1.1323 (0.3481)	1.9751 (0.7497)	0.0572 (0.0599)	0.0708 (0.1183)
60	0.5449 (0.1827)	0.7979 (0.4154)	0.0327 (0.0314)	0.0377 (0.0690)
	$(\alpha, \theta) = (2, 2)$			
	α		θ	
n	ML	Bootstrap	ML	Bootstrap
25	0.9288 (0.3589)	1.9856 (0.7901)	0.1992 (0.1419)	0.2772 (0.2821)
40	0.3777 (0.1896)	0.6149 (0.4046)	0.1131 (0.0807)	0.1400 (0.1634)
60	0.2047 (0.1204)	0.2897 (0.2511)	0.0684 (0.0521)	0.0799 (0.1064)

5. Real Data Application

In this section, we consider two real data applications in order to illustrate the adaptability of the ENXL distribution. We will analyze the fit of the ENXL distribution alongside several other lifetime distributions, which are:

1. The new XLindley (NXL) distribution, which is a special case of the ENXL distribution when α equals 1, (Khodja *et al.* , 2023).

2. The XLindley distribution with the following PDF (Chouia and Zeghdoudi , 2021)

$$f(x;\theta) = \frac{\theta^2}{(\theta+1)^2}(2+\theta+x)\,\mathrm{e}^{-\theta x}, \qquad x > 0, \quad \theta > 0.$$

3. The Lindley distribution with the following PDF (Lindley , 1958; Ghitany *et al.* , 2008)

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)\,\mathrm{e}^{-\theta x}, \qquad x > 0, \quad \theta > 0.$$

4. The exponential (Exp) distribution with PDF $f(x) = \theta\exp(-\theta x)$ where $x > 0$, and $\theta > 0$.

Table 2: The AWs and CPs of the 95% interval estimators.

		$(\alpha, \theta) = (2, 1)$					
		α			θ		
n		ACI	BBCI	PBCI	ACI	BBCI	PBCI
25	AW	2.9865	4.0786	4.0786	0.8006	0.8710	0.8710
	CP	0.9355	0.9000	0.8920	0.9255	0.9385	0.9030
40	AW	2.1344	2.5407	2.5407	0.6165	0.6481	0.6481
	CP	0.9520	0.9250	0.9230	0.9345	0.9410	0.9260
60	AW	1.6377	1.8343	1.8343	0.4977	0.5162	0.5162
	CP	0.9470	0.9175	0.9270	0.9365	0.9455	0.9290
		$(\alpha, \theta) = (3, 1.5)$					
		α			θ		
n		ACI	BBCI	PBCI	ACI	BBCI	PBCI
25	AW	5.2291	7.4783	7.4783	1.1127	1.2093	1.2093
	CP	0.9475	0.8985	0.8955	0.9320	0.9295	0.9115
40	AW	3.6487	4.4816	4.4816	0.8635	0.9102	0.9102
	CP	0.9390	0.9115	0.9080	0.9285	0.9375	0.9120
60	AW	2.7359	3.1176	3.1176	0.6926	0.7183	0.7183
	CP	0.9495	0.9115	0.9335	0.9450	0.9445	0.9340
		$(\alpha, \theta) = (2, 2)$					
		α			θ		
n		ACI	BBCI	PBCI	ACI	BBCI	PBCI
25	AW	3.0716	4.1684	4.1684	1.6080	1.7477	1.7477
	CP	0.9385	0.9130	0.8845	0.9250	0.9455	0.9010
40	AW	2.1336	2.5423	2.5423	1.2346	1.3020	1.3020
	CP	0.9450	0.9140	0.9140	0.9290	0.9400	0.9140
60	AW	1.6457	1.8424	1.8424	0.9934	1.0296	1.0296
	CP	0.9475	0.9200	0.9235	0.9420	0.9430	0.9275

5. The gamma distribution with the following PDF

$$f(x; \alpha, \theta) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\theta x}, \qquad x > 0, \quad \alpha, \theta > 0.$$

6. The Weibull distribution with the following PDF

$$f(x; \alpha, \theta) = \alpha\theta\, x^{\alpha-1} \mathrm{e}^{-\theta x^\alpha}, \qquad x > 0, \quad \alpha, \theta > 0.$$

7. The power Lindley (PL) distribution with the following PDF (Ghitany *et al.* , 2013)

$$f(x; \alpha, \theta) = \frac{\alpha\theta^2}{\theta+1} x^{\alpha-1}(1 + x^\alpha) \mathrm{e}^{-\theta x^\alpha}, \qquad x > 0, \quad \alpha, \theta > 0.$$

8. The exponentiated Rayleigh (E-R) distribution with the following PDF

$$f(x;\alpha,\theta) = 2\alpha\theta\, x\, \exp(-\theta x^2) \left[1 - \mathrm{e}^{-\theta x^2}\right]^{\alpha-1}, \quad x > 0, \quad \alpha, \theta > 0.$$

9. The modified weighted exponential (MWE) distribution with the following PDF (Chesneau *et al.* , 2022)

$$f(x;\alpha,\theta) = \theta \left(1 + \frac{1}{\alpha+1} - \exp(-\alpha\theta\, x)\right) \exp(-\theta x), \quad x > 0, \quad \alpha, \theta > 0.$$

10. The exponentiated XLindley (EXL) distribution with the following PDF (Alomair *et al.* , 2024)

$$f(x;\alpha,\theta) = \frac{\alpha\,\theta^2\,(2+\theta+x)}{(\theta+1)^2}\,\mathrm{e}^{-\theta x}\left[1 - (1 + \frac{\theta}{(1+\theta)^2})\mathrm{e}^{-\theta x}\right]^{\alpha-1}, \qquad x > 0,$$

where $\alpha > 0$ and $\theta > 0$.

"First, we analyze the annual rainfall data (measured in inches) documented at the Los Angeles Civic Center from 1999 to 2020, denoted as Data I (see www.laalmanac.com/weather/we08aa.php). The data are"

11.57	17.94	4.42	16.49	9.24	37.25	13.19	3.21	13.53	9.08	16.36
20.20	8.69	5.85	6.08	8.52	9.65	19.00	4.79	18.82	14.86	

Similar data sets were also analyzed by Asgharzadeh *et al.* (2015), Fallah *et al.* (2018), Khoshkhoo Amiri and MirMostafaee (2023) and Zanjiran and Mir-Mostafaee (2024).

Next, we consider a COVID-19 data set, that presents the daily number of deaths divided by daily new cases in the United States of America for 102 days, from 28 March to 7 July 2020, denoted by Data II, see Alsuhabi *et al.* (2022)

0.0149	0.0235	0.0230	0.0159	0.0200	0.0413	0.0360	0.0378
0.0363	0.0399	0.0453	0.0436	0.0598	0.0624	0.0546	0.0607
0.0609	0.0521	0.0615	0.0928	0.2232	0.0620	0.0812	0.0629
0.0651	0.0840	0.1072	0.0821	0.0567	0.0559	0.0606	0.0380
0.0586	0.0980	0.0925	0.0631	0.1869	0.0049	0.0176	0.0495
0.1112	0.0890	0.0940	0.0600	0.0652	0.0413	0.0588	0.0665
0.0816	0.0753	0.0579	0.0436	0.0527	0.0382	0.0568	0.0613
0.0531	0.0767	0.0400	0.0406	0.0237	0.0471	0.0722	0.0595
0.0597	0.0389	0.0265	0.0518	0.0419	0.0566	0.0516	0.0390
0.0245	0.0266	0.0314	0.0701	0.0410	0.0436	0.0320	0.0255
0.0171	0.0268	0.0259	0.0333	0.0318	0.0188	0.0172	0.0112
0.0155	0.0229	0.0184	0.0621	0.0146	0.0114	0.0216	0.0103
0.0129	0.0134	0.0117	0.0143	0.0032	0.0054		

We calculated the ML estimates (MLEs) of the parameters for the specified models. To assess the goodness of fit, we employed various measures, including the minus log-likelihood function evaluated at the ML estimates, denoted by $-\log \widehat{L}$, the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and the Kolmogorov-Smirnov (K-S) test statistic along with its corresponding p-value. These criteria allow us to compare the fit of the ENXL model with those of the Exp, NXL, XLindley, Lindley, Gamma, Weibull, PL, E-R, MWE, and EXL distributions.

We note that ties should not be present in a K-S test when analyzing continuous data. However, ties may occur as a result of rounding numbers or other reasons. To avoid this issue, we adjusted one of the equal numbers by adding (and also subtracting, in cases with three identical numbers) a very small number, $z = 10^{-14}$, when calculating the K-S test statistics. Thus, Data 2 has been changed to the following data to address this problem.

0.0149	0.0235	0.0230	0.0159	0.0200	0.0413	0.0360	0.0378
0.0363	0.0399	0.0453	0.0436	0.0598	0.0624	0.0546	0.0607
0.0609	0.0521	0.0615	0.0928	0.2232	0.0620	0.0812	0.0629
0.0651	0.0840	0.1072	0.0821	0.0567	0.0559	0.0606	0.0380
0.0586	0.0980	0.0925	0.0631	0.1869	0.0049	0.0176	0.0495
0.1112	0.0890	0.0940	0.0600	0.0652	0.0413+z	0.0588	0.0665
0.0816	0.0753	0.0579	0.0436+z	0.0527	0.0382	0.0568	0.0613
0.0531	0.0767	0.0400	0.0406	0.0237	0.0471	0.0722	0.0595
0.0597	0.0389	0.0265	0.0518	0.0419	0.0566	0.0516	0.0390
0.0245	0.0266	0.0314	0.0701	0.0410	0.0436-z	0.0320	0.0255
0.0171	0.0268	0.0259	0.0333	0.0318	0.0188	0.0172	0.0112
0.0155	0.0229	0.0184	0.0621	0.0146	0.0114	0.0216	0.0103
0.0129	0.0134	0.0117	0.0143	0.0032	0.0054		

The results are summarized in Tables 3 and 4 for Data I and Data II, respectively. From Tables 3 and 4, it is evident that the ENXL distribution outperforms the other considered models in terms of the measures considered, as this model shows the smallest AIC and BIC values and the largest p-value. The next best models are the Gamma and EXL distributions according to the same criteria.

Figures 4 and 5 display the empirical histograms of the data sets alongside the fitted PDFs of the considered models for Data I and Data II, respectively. Moreover, Figures 6 and 7 exhibit the probability-probability (P-P) plots for Data I and Data II, respectively.

From Figures 4–7, we may also conclude visually that the ENXL, Gamma, and EXL distributions offer the best fits for Data I and Data II among the distributions considered here. Nonetheless, upon reviewing Tables 3 and 4, we affirm the superiority of the ENXL distribution.

Table 3: The MLEs and goodness-of-fit criteria for Data I.

Distribution	MLE α	θ	$-\log\widehat{L}$	AIC	BIC	K-S	p-value
Exp		0.0781	74.5337	151.0673	152.1118	0.24804	0.1266
NXL		0.1224	72.9360	147.8719	148.9164	0.22570	0.2017
XLindley		0.1382	71.3069	144.6139	145.6584	0.17391	0.4951
Lindley		0.1463	70.6584	143.3168	144.3613	0.16178	0.5860
gamma	3.2250	0.2520	68.6944	141.3889	143.4779	0.09550	0.9809
Weibull	1.8212	0.0077	69.4212	142.8425	144.9315	0.11138	0.9314
PL	1.3199	0.0629	69.0449	142.0897	144.1788	0.10096	0.9683
E-R	0.9767	0.0045	69.5773	143.1547	145.2437	0.12588	0.8528
MWE	2.5758	0.0919	71.8310	147.6619	149.7510	0.19407	0.3605
EXL	2.4828	0.2078	68.6921	141.3843	143.4733	0.09203	0.9869
ENXL	2.9387	0.2039	**68.6676**	**141.3352**	**143.4243**	**0.09157**	**0.9875**

Table 4: The MLEs and goodness-of-fit criteria for Data 2.

Distribution	MLE α	θ	$-\log\widehat{L}$	AIC	BIC	K-S	p-value
Exp		20.4856	−206.0118	−410.0236	−407.3987	0.17855	0.0030
NXL		31.7806	−212.2505	−422.5011	−419.8761	0.15615	0.01383
XLindley		20.5298	−206.0119	−410.0239	−407.3989	0.17854	0.0030
Lindley		21.4002	−206.0674	−410.1349	−407.5099	0.17812	0.00309
gamma	2.3457	48.0524	−222.8574	−441.7147	−436.4648	0.08030	0.5263
Weibull	1.5788	98.6991	−221.2130	−438.4260	−433.1761	0.08929	0.3903
PL	1.5789	99.7155	−221.2107	−438.4214	−433.1714	0.08929	0.3902
E-R	0.7259	229.7007	−218.3239	−432.6479	−427.3979	0.11434	0.1389
MWE	2.4404	24.3338	−216.1339	−428.2679	−423.0179	0.13508	0.04835
EXL	2.5753	34.8563	−222.8236	−441.6471	−436.3972	0.08783	0.4109
ENXL	2.0270	45.0936	**−223.0781**	**−442.1561**	**−436.9062**	**0.07908**	**0.5462**

Concluding Remarks

In this paper, a new model, as an extension of the XLindley distribution, has been introduced, and some of its properties, such as moments, incomplete moments, mean deviations from the mean and the median, Bonferroni and Lorenz curves, mean residual life function, Rényi entropy, order statistics, and k-record values, have been discussed. The estimation of its parameters has been explored. The simulation study revealed that the ML method may perform better than the bootstrap method in point estimation. Additionally, if we use the delta method, the

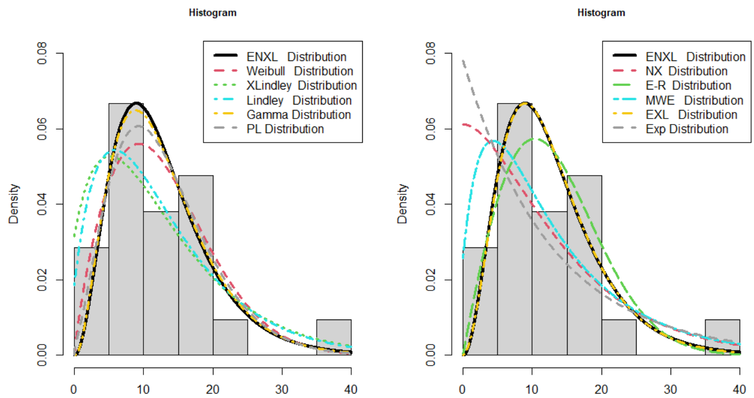

Figure 4: Empirical histogram of Data I and the fitted PDFs of the considered models.

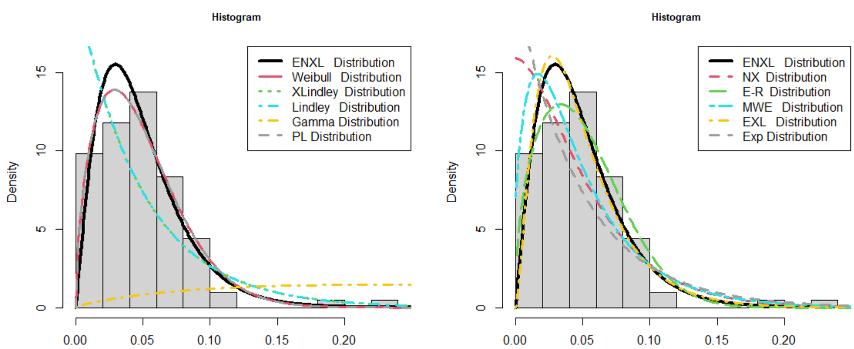

Figure 5: Empirical histogram of Data II and the fitted PDFs of the considered models.

ACIs also outperform the BBCIs and PBCIs in terms of AW. Two real data applications have verified the usefulness and superiority of the new model in comparison with some of the previously introduced models in real-world situations.

We suggest exploring Bayesian inference for the new distribution as a potential subject for future research, as this topic has not been addressed in the current paper. Furthermore, the application of linear inference to the exponentiated new XLindley distribution may present an interesting problem. Additionally, the task of introducing new extensions of the new XLindley distribution with more than two parameters, such as the exponentiated power new XLindley distribution with three parameters and the extended Marshall-Olkin generalized new XLindley distribution with four parameters (see Alizadeh *et al.* (2021)), might encourage us to explore this topic in the future.

All computations in this paper were performed using the statistical software

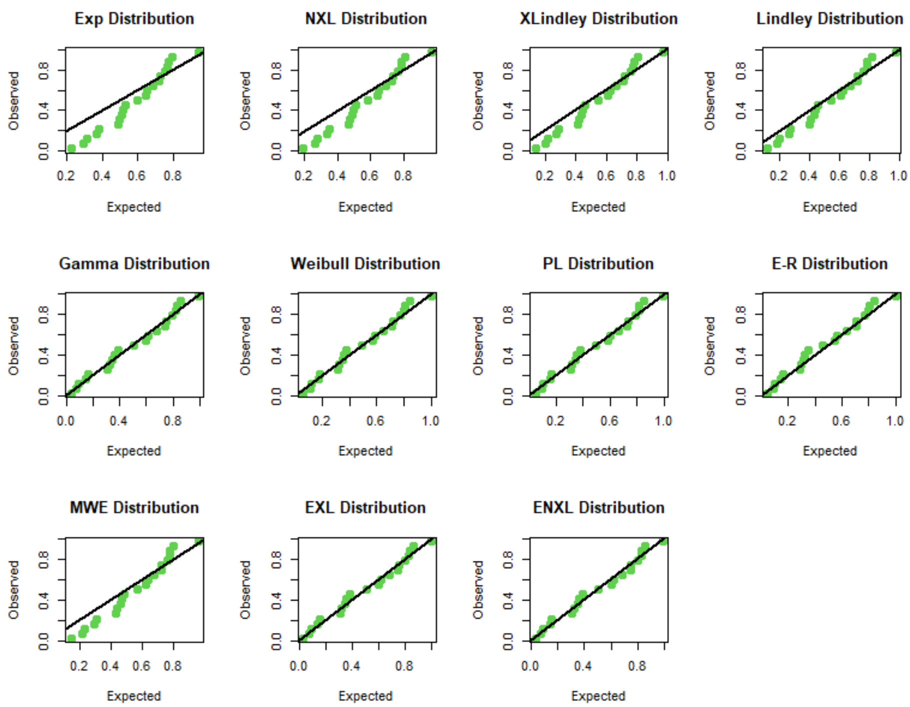

Figure 6: P-P plots for Data I.

R (R Core Team , 2024), along with the packages nleqslv (Hasselman , 2018) and AdequacyModel (Marinho *et al.* , 2016). The R codes related to the simulation study are provided as supplementary material. The R codes for plotting figures and analyzing the real data examples may be provided by the author upon request.

Acknowledgment

I would like to thank the two anonymous referees for their valuable comments.

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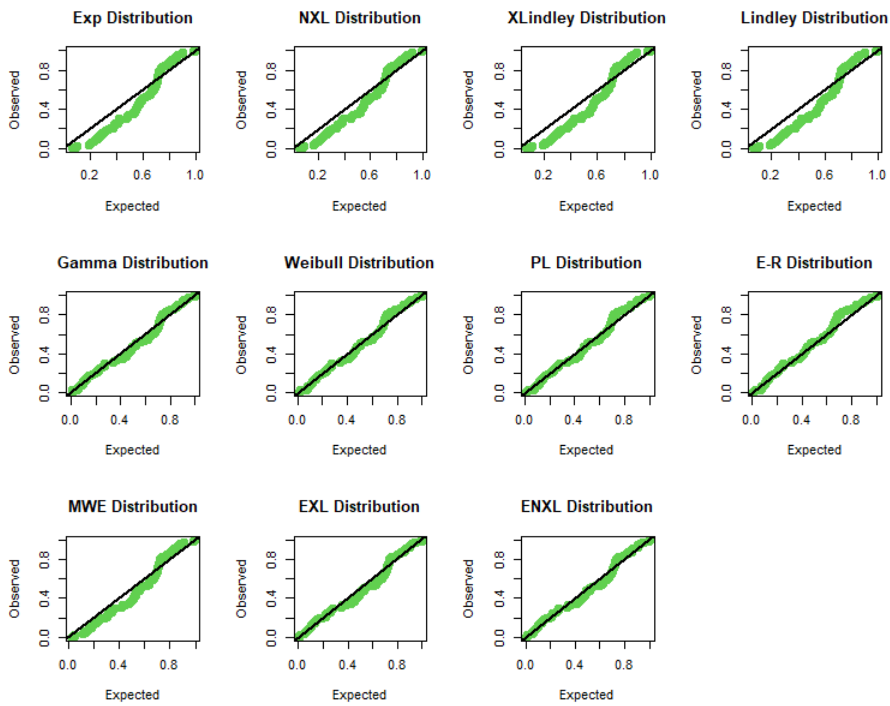

Figure 7: P-P plots for Data II.

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