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The Reliability Characteristics Estimation for a Family of Lifetime Distributions under Progressive Censoring

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Abstract: In this paper, the probability of failure-free operation until time t, along with the probability of stress-strength, based on progressive censoring data, is studied in a family of lifetime distributions. The classic maximum likelihood estimator (MLE) is proposed for unknown parameters. For a numerical demonstration of the proposed estimation strategies, some bootstrap confidence intervals, are constructed. The theoretical results are illustrated by a real data examples and an extensive simulation study. Simulation shreds of evidence revealed that our proposed strategies perform well in estimating parameters based on progressive censoring data. Finally, we applied the proposed methodology to estimate the probability of failure-free until time to a breakdown of an insulting fluid between electrodes and Stress-Strength reliability of the carbon fibers as well.

Keywords: Bootstrap; Lifetime; Progressive Censoring; Reliability; Stress-Strength.

Mathematics Subject Classification (2010): 62F10, 62G30.

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1. Introduction

The reliability function denoted by $R(t)$ is defined as the probability of failure-free operation until time t. Thus, if the random variable X denotes the lifetime of an item or system, then $R(t) = \mathbb{P}(X > t)$. Another measure of reliability under stress-strength set-up is the probability $\mathscr{P} = \mathbb{P}(X > Y)$, which represents the reliability of an item or system of random strength X subject to random stress Y, under a bivariate setting. These two reliability measures are frequently used in biostatistics, agriculture, engineering, and related fields. Hence, estimation is of importance. A lot of work has been done in the literature for the estimation and testing of the parameter, $R(t)$ and $\mathscr{P}$ under progressive censoring.

For a review of related studies, we refer to Pugh (1963), Basu (1964), Tong (1974), Tong (1975), Kelley et al (1976), Sathe and Shah (1981), Chao (1982), Awad and Gharraf (1986), Constantine et al (1986), Bhattacharya (1989), Chaturvedi and Rani (1997), Chaturvedi and Rani (1998), and Chaturvedi and Surinder (1999) among others. Baklizi (2003) proposed shrinkage estimators of $R(t)$ for one-parameter exponential distribution. Chaturvedi and Nandchahal (2016) and Chaturvedi and Shantanu (2017) estimated $R(t)$, by type-I and type-II censorings and in order to estimate $\mathscr{P}$, used complete sample case.

There are many scenarios in life-testing and reliability experiments in which units are lost or removed from the experimentation before failure. The damage may be unintentional, or it may be designed in the study. Unintentional damage may occur, for example, in the event of an unexpected breakdown of an experimental unit, or if an individual understudy drops out, or if the experimentation itself has to be stopped due to some unforeseen circumstances such as lack of funding, lack of access to testing facilities, etc. However, more often, units are removed from the preplanned and deliberate test, and to free up testing facilities for other experiments, to save time and money, or use a simple analysis that often results. Then in some cases, when there are live units on the test, the intentional deletion of the items or the end of the experiment may be due to ethical considerations. It has been used very effectively for analyzing lifetime data, especially when the data is censored. Among the various censoring schemes, the Type II progressive censoring scheme has become a very common design in the last decade because it allows the experimenter to omit active units during the experiment. So in this paper, we consider progressive Type-II censoring schemes. Under this scheme of censoring, from a total of n units placed on a life-test, only m are completely observed until failure. At the time of the first failure, R_1 of the $n-1$ surviving units are randomly withdrawn (or censored) from the life-testing experiment. At the time of the next

failure, R_2 out of the $n-2-R_1$ surviving units are censored, and so on. Finally, at the time of the m-th failure, all the remaining $R_m = n-m-R_1-...-R_{m-1}$ surviving units are censored. Note that censoring occurs here progressively in m stages. Clearly, this scheme includes as special cases the complete sample situation (when $m = n$ and $R_1 = ... = R_m = 0$) and the conventional Type-II right censoring situation (when $R_1 = ... = R_{m-1} = 0$ and $R_m = n-m$). In this censoring scheme, $R_1, R_2, \ldots, R_m$ (and therefore m) are all prefixed. Consequently, here the censoring times (T's) are all random, but the numbers of items to fail before each censoring time are all fixed. For more details see Balakrishnan et al (2000).
Suppose n independent units are placed on a life-test with the corresponding failure times $X_1, X_2, ..., X_n$ being identically distributed with cumulative distribution function $F(x)$ and probability density function $f(x)$. Suppose further that the prefixed number of failures to be observed is m and that the progressive Type-II right censoring scheme is $(R_1, R_2, \ldots, R_m)$. Then, we shall denote the m completely observed failure times by $X_{i:m:n}^{(R_1,R_2,\ldots,R_m)}, i = 1, 2, \ldots, m$. For simplicity in notation, when it is clear as to what the censoring scheme is, we will use the simplified notation $X_{i:m:n}, i = 1, 2, \ldots, m$, to denote these failure times bearing in mind that these still depend on the particular choice of $(R_1, R_2, \ldots, R_m)$ units. we can write down the joint probability density function (pdf) of all m progressively Type-II right censored order statistics as

$$f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(\boldsymbol{x}) = c\prod_{i=1}^{m} f(x_i)\{1-F(x_i)\}^{R_i}, \; x_1 < x_2 < ... < x_m, \quad (1.1)$$

where

$$\boldsymbol{x} = x_1, x_2, ..., x_m, \quad c = n(n-R_1-1)\ldots(n-R_1-R_2-...-R_{m-1}-m+1).$$

In sequel, we employ a family of lifetime distributions for the model of $f(.)$ in (1.1). Let the random variable X follows a distribution having the pdf

$$f(x; a, \lambda, \boldsymbol{\theta}) = \lambda G'(x; a, \boldsymbol{\theta}) exp\{-\lambda G(x; a, \boldsymbol{\theta})\} \; ; x > a \geq 0, \; \lambda > 0. \quad (1.2)$$

Here, $G(x; a, \boldsymbol{\theta})$ is a function of x and may also depend on the (known) parameters a and $\boldsymbol{\theta}$ may be vector valued. Moreover, $G(x; a, \boldsymbol{\theta})$ is monotonically increasing in x with $G(a; a, \boldsymbol{\theta}) = 0$; $G(\infty; a, \boldsymbol{\theta}) = \infty$ and $G'(x; a, \boldsymbol{\theta})$ denotes the derivative of $G(x; a, \boldsymbol{\theta})$ with respect to x. For more details see Chaturvedi and Nandchahal (2016). We call this family of lifetime distribution as $\mathcal{CN}$ with mentioned parameters and designate $X \sim \mathcal{CN}(a, \lambda, \boldsymbol{\theta}, G)$. Hence

$$R(t) = exp(-\lambda G(t; a, \boldsymbol{\theta})) \quad (1.3)$$

and the hazard rate has form $h(t) = \lambda G'(t; a, \boldsymbol{\theta})$.
Let $S_m = \sum_{i=1}^{m} G(X_i; a, \boldsymbol{\theta}) + \sum_{i=1}^{m} R_i G(X_i; a, \boldsymbol{\theta})$. Then the likelihood function is given by

$$L(\lambda | x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}; a; \boldsymbol{\theta}) = c\lambda^m \prod_{i=1}^{m} G'(X_i; a, \boldsymbol{\theta}) \exp(-\lambda S_m) \tag{1.4}$$

The maximum likelihood estimators (MLEs) of λ and $R(t)$ are, respectively, given by

$$\hat{\lambda} = \frac{m}{S_m}, \tag{1.5}$$

$$\hat{R}(t) = \exp\left\{ -\frac{m}{S_m} G(t; a, \boldsymbol{\theta}) \right\}. \tag{1.6}$$

For the estimation of λ, we use the likelihood method. Let $X \sim \mathcal{CN}(a, \lambda_1, \boldsymbol{\theta_1}, G)$ is independent of $Y \sim \mathcal{CN}(a, \lambda_2, \boldsymbol{\theta_2}, G)$. Then, we have

$$\mathscr{P} = \frac{\lambda_1}{\lambda_1 + \lambda_2}. \tag{1.7}$$

The MLE of $\mathscr{P}$ has form

$$\hat{\mathscr{P}} = \frac{\hat{\lambda}_1}{\hat{\lambda}_1 + \hat{\lambda}_2}, \tag{1.8}$$

where $\hat{\lambda}_1 = \frac{m_1}{S_{m_1}}$ and $\hat{\lambda}_2 = \frac{m_2}{T_{m_2}}$, where

$$S_{m_1} = \sum_{i=1}^{m_1} G(X_i; a, \boldsymbol{\theta_1}) + \sum_{i=1}^{m_1} R_i G(X_i; a, \boldsymbol{\theta_1})$$
$$T_{m_2} = \sum_{i=1}^{m_2} G(Y_i; a, \boldsymbol{\theta_2}) + \sum_{i=1}^{m_2} R_i G(Y_i; a, \boldsymbol{\theta_2}).$$

The purpose of this study is to provide MLE for proposed parameters. Therefore, we organize our paper as follows. We also derive exact formulas for the properties of estimators in this section. Section 2 contains an extensive simulation study to evaluate the performance of the proposed estimators. Section 3 is devoted to the estimation involving real situations, numerically. We conclude our paper in Section 4.

1.0.1 Small sample properties

In this section, we derive the bias, variance, and MSE of the proposed estimators. For the notation convenience, let

- $\varphi_4 = \mathrm{E}(\hat{R}(t)) = \frac{2(m\lambda G(t;a,\boldsymbol{\theta}))^{\frac{m}{2}}}{\Gamma(m)} K_m(2\sqrt{m\lambda G(t;a,\boldsymbol{\theta})})$
- $\varphi_5 = \mathrm{E}(\hat{R}^2(t)) = \frac{2(2m\lambda G(t;a,\boldsymbol{\theta}))^{\frac{m}{2}}}{\Gamma(m)} K_m(2\sqrt{2m\lambda G(t;a,\boldsymbol{\theta})})$

where $K_r(.)$ is the modified Bessel function of the second kind of order r. (see Watson (1995))
The following theorem gives the bias expression of the estimator of $R(t)$:

Theorem 1.1. *The bias expressions for the estimator is given by*

$$\mathrm{Bias}(\hat{R}(t)) = \varphi_4 - R(t)$$

For the proof, refer to Appendix.
The next Theorem gives the MSE expression.

Theorem 1.2. *The MSE expressions of the estimator is given by*

$$\mathrm{MSE}(\hat{R}(t)) = \varphi_5 - 2R(t)\varphi_4 + R^2(t)$$

For the proof refer to Appendix.
The following result will be used in the algorithm of bootstrap confidence interval (CI).

Corollary 1.3. *The variance expressions of the estimator is given by*

$$\mathrm{Var}(\hat{R}(t)) = \varphi_5 - (\varphi_4)^2 \tag{1.9}$$

1.1 Estimation of $\mathscr{P}$

As we estimated MLE for $\mathscr{P}$ in formula 1.8 we try asses this estimator in small samples.
If we define statistic $\mathrm{L} = \frac{S_{m_1}}{T_{m_2}}$, then it follows the F-distribution with $(2m_1, 2m_2)$ degrees of freedom and having the pdf

$$f(F) = \frac{\left(\frac{m_1}{m_2}\right)^{m_1}}{B(m_1, m_2)} \cdot \frac{\mathrm{F}^{m_1-1}}{\left[1 + \frac{m_1}{m_2}\mathrm{F}\right]^{m_1+m_2}};\ 0 < \mathrm{F} < \infty.$$

Making the transformation

$$\mathrm{W} = \left(1 + \frac{\lambda_2}{\lambda_1}\mathrm{F}\right)^{-1},$$

the pdf of W comes out to be

$$f(w) = \frac{\left(\frac{m_2\lambda_2}{m_1\lambda_1}\right)^{m_2}}{\mathrm{B}(m_1, m_2)} \cdot \frac{w^{m_2-1}(1-w)^{m_1-1}}{\left[1 + \left(\frac{m_2\lambda_2}{m_1\lambda_1} - 1\right) w\right]^{m_1+m_2}};\ 0 < w < 1.$$

1.1.1 Small sample properties

For the ease of use, let

- $\psi_1 \quad = \quad \mathrm{E}(\hat{\mathscr{P}}) = \mathrm{E}\left(1 + \frac{\hat{\lambda_2}}{\hat{\lambda_1}}\right)^{-1} = \mathrm{E}\left(1 + \frac{\lambda_2}{\lambda_1}\mathrm{F}\right)^{-1} = \mathrm{E}(\mathrm{W}),$
- $\psi_2 \quad = \quad \mathrm{E}(\hat{\mathscr{P}}^2) = \mathrm{E}(\mathrm{W}^2).$

The following theorem gives the bias expression of the estimator of the $\mathscr{P}$:

Theorem 1.4. *The bias expressions for the unrestricted, the estimator is given by*

$$\mathrm{Bias}(\hat{\mathscr{P}}) = \psi_1 - \mathscr{P}$$

For the proof refer to Appendix.

Theorem 1.5. *The MSE expressions for ML estimator are given by*

$$\mathrm{MSE}(\hat{\mathscr{P}}) = \psi_2 - 2\mathscr{P}\psi_1 + \mathscr{P}^2$$

,

The proof is similar to Theorem 1.2.

Corollary 1.6. *The variance expression of the estimator is given by*

$$\mathrm{Var}(\hat{\mathscr{P}}) = \psi_2 - \psi_1^2 \tag{1.10}$$

2. Simulation Study

Here we conduct a Monte Carlo simulation study with a small sample size to assess the performance of meteorologies developed in this paper.

The simulation composition and assumptions are as follows:

$R(t)$: The true value of reliability is taken to be $\{0.50, 0.55, 0.60, 0.65, 0.70\}$.

m: number of observations is taken to be 10.

$R = (R_1, \ldots, R_m)$: progressive Type-II censoring scheme and is taken to be $R = (25, 10, 7, 5, 3, 10, 9, 5, 7, 9)$ and $R = (5, 4, 3, 2, 1, 1, 1, 1, 1, 71)$.

t: truncation time point which is equal to 3.

For each combination of R, 1000 samples of size 50, 100 and 200 were generated from the distribution given in (1.2), taking $G(x;a;\boldsymbol{\theta}) = x$. The proposed estimators for $R(t)$ are calculated under progressive Type-II censoring and their CIs are computed. let (L, U) be a CI of $R(t)$ and (L_i, U_i), $i = 1, 2, \ldots, 1000$, observed values of lower and upper bounds of the proposed CI. Thus average of expected length and coverage probability are respectively, given by

$$EL = \frac{1}{1000}\sum_{i=1}^{1000}(U_i - L_i) \qquad CP = \frac{1}{1000}\sum_{i=1}^{1000} I(L_i \leq R \leq U_i),$$

Tables 1 represents the coverage probability (CP) and expected length (EL) of the estimators for $R(t)$ under progressive Type-II censoring scheme from an exponential model. The Algorithm 1 is used to obtaining bootstrap-t CIs based on the idea of Efron (1982).

According to this Table 1 we can see that the proposed estimator almost has good CP and short ELs, so it has good performance. Furthermore when the $R(t)$ is close to 1, it performs better than when it is close to 0.5. Because the CPs of the estimator are getting higher and Els are getting shorter values by approaching the $R(t)$ to 1. Another thing is, that there is no evidence of different performance between the two mentioned progressive censoring schemes. Their performances are almost the same.

The indices of the simulation for for the $\mathscr{P}$ are as follows:

$\mathscr{P}$: the true value of $\mathscr{P} = \mathbb{P}(X > Y)$ are taken to be $\{.5, .55, .58, 0.6, .63, 0.65, .68, .7, .73, .75, .78, .8, .85, .9\}$

m_1: number of X observations is taken to be 10.

m_2: number of Y observations is taken to be 8.

$R = (R_1, \ldots, R_{m_1})$: progressive Type-II censoring scheme for X is taken to be $R = (25, 10, 7, 5, 3, 10, 9, 5, 7, 9)$.

$R' = (R'_1, \ldots, R'_{m_2})$: progressive Type-II censoring scheme Y is taken to be $R' = (20, 10, 5, 5, 10, 10, 9, 43)$.

For each combination of $\mathscr{P}$, 1000 samples of size $n_1 = 100$ were generated for X from the distribution given in (1.2), taking $\lambda_1 = 0.3$ and $G(x;a_1;\theta_1) = x$ and 1000 samples of size $n_2 = 120$ were generated for Y from the same distribution with $\lambda_2 = \frac{1}{\mathscr{P}} - 1$ and $G(y;a_2;\theta_2) = y$. The proposed estimator for $\mathscr{P}$ is calculated under progressive Type-II censoring and their CIs are computed. The Algorithm 2 is used to obtaining bootstrap-t CIs by proposed estimators for $\mathscr{P}$.

The results of simulation presented in Table 2. As one can see, unlike the ML estimator for the $R(t)$, the MLE of $\mathscr{P}$ has very high CPs close to 1 and short ELs, so it has very good performance in both of the CI algorithms. Further more, the ELs are getting shorter when the $\mathscr{P}$ is closer to 1.

Algorithm 1 Bootstrap-t CI for $R(t)$ based on the bootstrap variance estimate

Step 1. Based on the independent observed samples $X_{i:m:n}^{(R_1,R_2,\ldots R_m)}, i = 1,...,m$ with progressive Type-II right censoring scheme $(R_1, R_2, \ldots, R_m)$, $\hat{\lambda}$ and $\hat{R}(t)$ estimators from (1.5) and (1.6) respectively.

Step 2. Generate $X_{i:m:n}^{\star} \sim \text{Exp}(\hat{\lambda}), i = 1,...,m$. Use them to obtain $\hat{\lambda}^{\star}$ and $\hat{R}^{\star}(t)$.

Step 3. Repeat Step 2 for B times and derive $\hat{R}_{(b)}^{\star}(t)$, $b = 1, \ldots, B$.

Step 4. For each iteration of Step 3, design another parametric bootstrap procedure to estimate the standard deviation of $\hat{R}_{(b)}^{\star}(t)$, say $\hat{\sigma}(\hat{R}_{(b)}^{\star}(t))$. More precisely, repeat Step 2 for $b' = 1, \ldots, B''$, with $\hat{\lambda}^{\star}$ instead of $\hat{\lambda}$, and then calculate

$$\hat{\sigma}(\hat{R}_{(b)}^{\star}(t)) = \sqrt{\frac{1}{B'-1} \sum_{b=1}^{B'} (\hat{R}_{(b')}^{\star\star}(t) - \bar{R}^{\star\star}(t))^2}$$

where $\bar{R}^{\star\star}(t) = \frac{1}{B'} \sum_{b'=1}^{B'} \hat{R}_{(b')}^{\star\star}(t)$.

Step 5. Let $t^{\star} = \left(t_{(1)}^{\star}, \ldots, t_{(B)}^{\star}\right)^{\top}$, where $t_{(b)}^{\star} = \frac{\hat{R}_{(b)}^{\star}(t) - \hat{R}(t)}{\hat{\sigma}(\hat{R}_{(b)}^{\star}(t))}, b = 1, \ldots, B$.

Step 6. Compute the $100(1-\alpha)\%$ bootstrap-t CI for $R(t)$ as $\left(\hat{R}(t) - t_{1-\frac{\alpha}{2}}^{\star}\hat{\sigma}(\hat{R}(t)), \hat{R}(t) - t_{\frac{\alpha}{2}}^{\star}\hat{\sigma}(\hat{R}(t))\right)$, where $t_{\gamma}^{\star}$ is $100\gamma\%$th percentile of $t^{\star}$ given by Step 5 and $\hat{\sigma}(\hat{R}(t)) = \sqrt{\hat{\text{Var}}(\hat{R}(t))}$ given by (1.9).

Table 1: The CP and EL of $R(t)$.

		$R = (25,10,7,5,3,10,9,5,7)$			$R = (5,4,3,2,1,1,1,1,1)$		
	$R(t)$	n	CP.$\hat{R}(t)$	EL.$\hat{R}(t)$	n	CP.$\hat{R}(t)$	EL.$\hat{R}(t)$
1	0.55	100	0.86	0.34	50	0.87	0.35
2	0.55	200	0.88	0.34	100	0.86	0.34
3	0.60	100	0.86	0.33	50	0.88	0.33
4	0.60	200	0.86	0.33	100	0.88	0.32
5	0.65	100	0.87	0.32	50	0.88	0.32
6	0.65	200	0.88	0.31	100	0.90	0.31
7	0.70	100	0.91	0.28	50	0.92	0.28
8	0.70	200	0.90	0.29	100	0.88	0.28

Algorithm 2 Bootstrap-t CI for $\mathscr{P}$ based on the bootstrap variance estimate

Step 1. Based on the independent observed samples $X_{i:m_1:n_1}^{(R_1,R_2,\dots R_{m_1})}, i=1,...,m_1$ with progressive Type-II right censoring scheme $(R_1, R_2, \dots, R_{m_1})$ and $Y_{j:m_2:n_2}^{(R'_1,R'_2,\dots R'_{m_2})}, j=1,...,m_2$ with progressive Type-II right censoring scheme $(R'_1, R'_2, \dots, R'_{m_2})$, $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\mathscr{P}}$ from (1.5) and (1.8) respectively.

Step 2. Generate $X^{\star}_{i:m_1:n_1} \sim \mathrm{Exp}(\hat{\lambda}_1), i=1,...,m_1$ and $Y^{\star}_{j:m_2:n_2} \sim \mathrm{Exp}(\hat{\lambda}_2), j = 1,...,m_2$. Use them to obtain $\hat{\lambda}^{\star}_1$, $\hat{\lambda}^{\star}_2$ and $\hat{\mathscr{P}}^{\star}$.

Step 3. Repeat Step 2 for B times and derive $\hat{\mathscr{P}}^{\star}_{(b)}$, $b=1,\dots,B$.

Step 4. For each iteration of Step 3, design another parametric bootstrap procedure to estimate the standard deviation of $\hat{R}^{\star}_{(b)}(t)$, say $\hat{\sigma}(\hat{\mathscr{P}}^{\star}_{(b)}(t))$. More precisely, repeat Step 2 for $b'=1,\dots,B''$, with $\hat{\lambda}^{\star}_1$ and $\hat{\lambda}^{\star}_2$ instead of $\hat{\lambda}_1$ and $\hat{\lambda}_2$, and then calculate

$$\hat{\sigma}(\hat{\mathscr{P}}^{\star}_{(b)}) = \sqrt{\frac{1}{B'-1}\sum_{b=1}^{B'}(\hat{\mathscr{P}}^{\star\star}_{(b')} - \bar{\mathscr{P}}^{\star\star})^2}$$

where $\bar{\mathscr{P}}^{\star\star} = \frac{1}{B'}\sum_{b'=1}^{B'} \hat{\mathscr{P}}^{\star\star}_{(b')}$.

Step 5. Let $t^{\star} = \left(t^{\star}_{(1)}, \dots, t^{\star}_{(B)}\right)^{\top}$, where $t^{\star}_{(b)} = \frac{\hat{\mathscr{P}}^{\star}_{(b)} - \hat{\mathscr{P}}}{\hat{\sigma}(\hat{\mathscr{P}}^{\star}_{(b)})}, b=1,\dots,B$.

Step 6. Compute the $100(1-\alpha)\%$ bootstrap-t CI for $\mathscr{P}$ as $\left(\hat{\mathscr{P}} - t^{\star}_{1-\frac{\alpha}{2}}\hat{\sigma}(\hat{\mathscr{P}}), \hat{\mathscr{P}} - t^{\star}_{\frac{\alpha}{2}}\hat{\sigma}(\hat{\mathscr{P}})\right)$, where $t^{\star}_{\gamma}$ is $100\gamma\%$th percentile of $t^{\star}$ given by Step 5 and $\hat{\sigma}(\hat{\mathscr{P}}) = \sqrt{\hat{\mathrm{Var}}(\hat{\mathscr{P}})}$ given by (1.10).

Table 2: The CP and EL of $\mathscr{P}$.

		$\hat{\mathscr{P}}$			
		Boot.		Asymp.	
	$\mathscr{P}$	CP	EL	CP	EL
1	0.50	0.99	0.56	0.95	0.45
2	0.55	1.00	0.56	0.95	0.45
3	0.58	0.99	0.55	0.96	0.44
4	0.60	1.00	0.54	0.95	0.44
5	0.63	0.99	0.52	0.96	0.43
6	0.65	0.99	0.51	0.96	0.42
7	0.68	0.98	0.49	0.94	0.40
8	0.70	0.99	0.47	0.94	0.39
9	0.73	0.98	0.45	0.95	0.37
10	0.75	0.98	0.42	0.95	0.35
11	0.78	0.98	0.39	0.95	0.33
12	0.80	0.97	0.36	0.94	0.31
13	0.85	0.95	0.29	0.94	0.25
14	0.90	0.93	0.21	0.93	0.18

3. Examples On Real Data

In this section, we analyze the performance of the proposed estimators using two real data sets. The first one is the estimation of $R(t)$ and the second example is for $\mathscr{P}$.

3.1 Time to the breakdown of an insulating fluid between electrodes

Here, we consider the real data set used in Lawless (1982). These data are from Nelson and Winter (1982), concerning the time to the breakdown of an insulating fluid between electrodes at a voltage of 34 kV (minutes). The 19 times to breakdown are

0.96 4.15 0.19 0.78 8.01 31.75 7.35 6.50 8.27 33.91 32.52 3.16 4.85 2.78 4.67 1.31 12.06 36.71 72.89

Therefore, we observe 9 progressively censoring values under scheme $(R_1 = 2, R_2 = 2, R_3 = 0, R_4 = 0, R_5 = 0, R_6 = 0, R_7 = 1, R_8 = 1, R_9 = 4)$ as

0.19 0.78 1.31 2.78 4.15 4.67 4.85 6.50 8.01

Table 3: Estimators of $R(t)$ in time to breakdown data .

	Estimated value	Variance	Bootstrap CI B=200	Asymp. CI
$\hat{R}(t)$	0.7041	0.0067	(0.59490, 0.8605)	(0.5884, 0.9092)

Chaturvedi and Malhotra (2017) applied the Kolmogorov-Smirnov (K-S) test as well as the Chi-Square test to show that the Weibull distribution is a suitable for the fixed voltage level, time to breakdown data. The MLEs of the parameters of Weibull distribution are as $\hat{p} = 0.7708$, $\hat{\lambda} = 6.8865$. Hence $\hat{R}(t)|_{t=2} = 0.7488$.

3.2 Stress-Strength of the carbon fibers

In this part, we analyze the data reported by Bader and Priest (1982) . This data represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 20mm (Data Set 1) and 10mm (Data Set 2) with sample sizes 69 and 63, respectively. These data have been used previously by Raqab and Kundu (2005), Kundu and Gupta (2006), Kundu and Raqab (2009), Asgharzadeh et al (2011) and Chaturvedi and Nandchahal (2016). Kundu and Gupta (2006) analyzed these data sets using two-parameter Weibull distribution after subtracting 0.75 from both these data sets. After subtracting 0.75 from all the points of these data sets, Kundu and Gupta (2006) observed that the Weibull distributions with equal shape parameters fit to both the data sets. The MLEs of the parameters of Weibull distribution fitting data set 1 are $\hat{\lambda}_1 = 0.0046$ and $\hat{p}_1 = 5.5049$ respectively. Similarly, for the data set 2, $\hat{\lambda}_2 = 0.0023$ and $\hat{p}_2 = 5.0494$. For comparing results, we have used two different progressively censored samples using two different sampling schemes tabulated in Tables 4 and 5, generated by Asgharzadeh et al (2011). The generated data and corresponding censored schemes have been presented in Table 6. The point and interval estimation for the $\mathscr{P}$ are calculated and presented in Table 7. By the estimated value of $\mathscr{P}$ as 0.1767, we conclude that the carbon fiber with gauge length 20mm has less strength than gauge length 10mm.

4. Conclusion

In this paper, we discussed reliability characteristics such as $R(t)$ and $\mathscr{P}$ parameters in progressively censored samples of a family of lifetime distributions. We

Table 4: Data Set 1 (gauge length of 20 mm):

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	

Table 5: Data Set 2 (gauge length of 10 mm):

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	4.395	5.020							

Table 6: Data and the corresponding censored schemes.

i, j	1	2	3	4	5	6	7	8	9	10
x_i	1.312	1.479	1.552	1.803	1.944	1.858	1.966	2.027	2.055	2.098
R_i	1	0	1	2	0	0	3	0	1	50
y_j	1.901	2.132	2.257	2.361	2.396	2.445	2.373	2.525	2.532	2.575
R'_j	0	2	1	0	1	1	2	0	0	44

Table 7: Estimators of $\mathscr{P}$ for gauge data .

$\alpha = 0.05$	Estimated value	Asymp. CI	Boot. CI
$\hat{\mathscr{P}}$	0.1767	(0.0156 0.3377)	(0.1311 0.2535)

estimated the ML for these parameters and derived distributional properties of them. The numerical analysis showed that the proposed estimators have good performances and are better when the parameters are close to 1.

Appendix

In this section, we provide the sketch of proofs of theorems.

Proof of Theorem 1.1:

$$\begin{aligned}
\text{Bias}(\hat{R}(t)) &= \mathrm{E}\left(\hat{R}(t) - R(t)\right) = \mathrm{E}(e^{-\frac{m}{S_m}G(t;a,\boldsymbol{\theta})}) - R(t) \\
&= \int_0^\infty \exp\{\frac{-mG(t;a,\boldsymbol{\theta})}{S_m}\}\frac{\lambda^m S_m^{m-1}e^{-\lambda S_m}}{\Gamma(m)}ds_m - R(t) \\
&= \frac{1}{\Gamma(m)}\int_0^\infty w^{m-1}\exp\{-(\frac{2m\lambda G(t;a,\boldsymbol{\theta})}{w} + \frac{w}{2})\}dw - R(t) \\
&= \frac{2}{\Gamma(m)}\{m\lambda G(t;a,\boldsymbol{\theta})\}^{\frac{m}{2}}\mathrm{K}_m(2\sqrt{m\lambda G(t;a,\boldsymbol{\theta})}) - R(t) = \varphi_4 - R(t),
\end{aligned}$$

Proof of Theorem 1.2:

$$\begin{aligned}
\text{MSE}(\hat{R}(t)) &= \mathrm{E}\left(\hat{R}(t) - R(t)\right)^2 = \mathrm{E}\left(\hat{R}^2\right) - 2R(t)\mathrm{E}(\hat{R}) + R^2(t) \\
&= \mathrm{E}\left(\exp(-\frac{4m\lambda G(t;a,\boldsymbol{\theta})}{w})\right) - 2R(t)\mathrm{E}\left(\exp(-\frac{2m\lambda G(t;a,\boldsymbol{\theta})}{w})\right) + R^2(t) \\
&= \frac{2(2m\lambda G(t;a,\boldsymbol{\theta}))^{\frac{m}{2}}}{\Gamma(m)}K_m(2\sqrt{2m\lambda G(t;a,\boldsymbol{\theta})}) \\
&\quad -4R(t)\frac{(m\lambda G(t;a,\boldsymbol{\theta}))^{\frac{m}{2}}}{\Gamma(m)}K_m(2\sqrt{m\lambda G(t;a,\boldsymbol{\theta})}) + R^2(t) \\
&= \varphi_5 - 2R(t)\varphi_4 + R^2(t).
\end{aligned}$$

Proof of Theorem 1.4:

$$\text{Bias}(\hat{\mathscr{P}}) = \mathrm{E}(\hat{\mathscr{P}}) - \mathscr{P} = \mathrm{E}(\mathrm{W}) - \mathscr{P} = \psi_1 - \mathscr{P}.$$

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